15.4: Apps of Double Integrals

(I) Physics

- mass
- moments
- center of mass
- moments of inertia
- radius of inertia

(II) Probability: Joint density fcts of 2 variables.

If $\rho(x,y)$ gives the density of a lamina at a pt (x,y), why does $\rho(x,y)\,dA$ give the mass of the infinitesimal rectangular region?

Mass: $m = \iint_D \rho(x,y)\,dA$

<u>ex1</u>: Find the following of a lamina w/ uniform density 1 enclosed by the cardiod $r = 1+\cos\theta$.

(a) The mass

$$m = 2\int_0^{\pi}\int_0^{1+\cos\theta} 1\cdot r\,dr\,d\theta$$

$$\vdots$$

$$= \frac{3}{2}\pi \quad (\text{from } 15.3)$$

The moment of a particle is the product of its mass & dist. from the axis.

$$m_x = \iint_D \underbrace{y}_{\text{DIST}}\, \underbrace{\rho(x,y)\,dA}_{\text{mass}} \quad \& \quad m_y = \iint_D \underbrace{x}_{\text{DIST}}\, \underbrace{\rho(x,y)\,dA}_{\text{MASS}}$$

(b) Find the moments.

$$m_x = \int_0^{2\pi}\int_0^{1+\cos\theta} y \cdot 1 \; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \sin\theta \,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}\Big[r^3 \sin\theta\Big]_0^{1+\cos\theta} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} (1+\cos\theta)^3 \sin\theta \; d\theta$$

Let $u = 1+\cos\theta$
$du = -\sin\theta\, d\theta$
$u(0) = 2$
$u(2\pi) = 2$

$$= -\frac{1}{3}\int_2^2 u^3\, du$$

$$= 0$$

$$m_y = \int_0^{2\pi}\int_0^{1+\cos\theta} x \cdot 1 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \cos\theta\, dr\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \Big[r^3\cos\theta\Big]_0^{1+\cos\theta} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} (1+\cos\theta)^3 \cos\theta\; d\theta$$

$$= \frac{5\pi}{4} \quad \text{(via Mathematica)}$$

The center of mass is at $\bar{x} = \frac{M_y}{m}$ & $\bar{y} = \frac{M_x}{m}$

(c) Find the centroid.

$$\bar{x} = \frac{5\pi}{4} \cdot \frac{2}{3\pi} = \frac{5}{6} \quad \& \quad \bar{y} = 0$$

$$\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{5}{6}, 0\right)$$

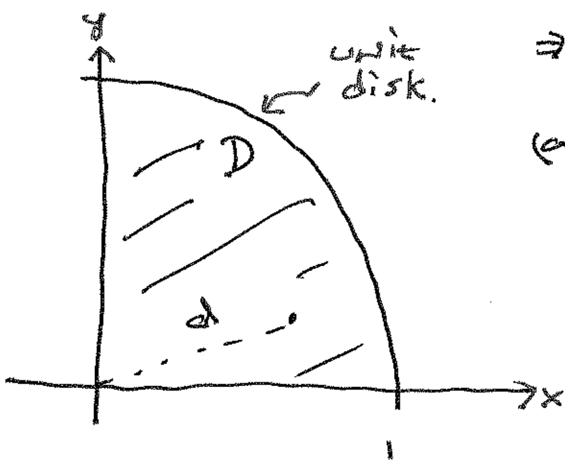
see pic on page 4.

Note that $\bar{y}$ is obvious given the graph. //

After ex 2: The moment of inertia are calculated in a similar manner.

$$I_x = \iint_D y^2 \rho(x,y)\,dA \quad \text{and} \quad I_y = \iint_D x^2 \rho(x,y)\,dA.$$

ex2: A lamina occupies the part of the unit circle in Q1. The density @ any pt is proportional to the square of the dist. from the origin. $d = \sqrt{x^2 + y^2}$



$$\Rightarrow \rho(x,y) = k(x^2 + y^2)$$

(a) $m = k\iint_D (x^2 + y^2)\,dA$

$$= k\int_0^{\pi/2}\int_0^1 r^2 \cdot r\,dr\,d\theta$$

$$= k\frac{1}{4} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{8}k$$

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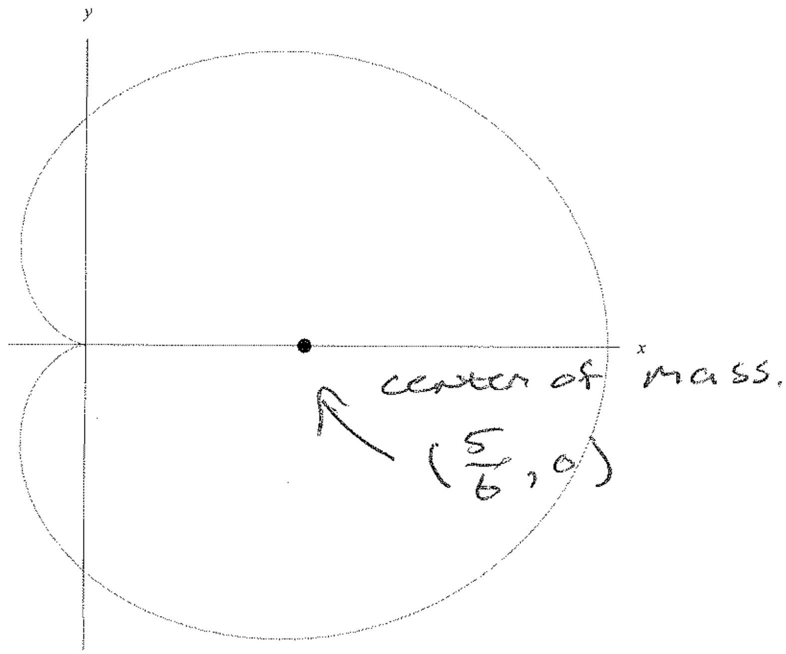
In[1]:= Show[PolarPlot[1 + Cos[θ], {θ, 0, 2 π}, AspectRatio → 1],
    Graphics[{PointSize → Large, Point[{5/6, 0}]}], Ticks → None, AxesLabel → {x, y}]

Out[1]=



Note: we could determine $\bar{y} = 0$ in this case by symmetry. But, if ρ had been non-constant, then ... we would have had to work.

(b) $m_x = \iint_D y(x^2+y^2)\,dA$

$= \int_0^{\pi/2}\int_0^1 r\sin\theta \cdot r^2 \cdot r\,dr\,d\theta$

$= \int_0^{\pi/2} \sin\theta\,d\theta \cdot \int_0^1 r^4\,dr$

$= \frac{1}{5}$

$m_y = \frac{1}{5}$ (by symmetry)

so $(\bar{x}, \bar{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$

↳ moment of inertia formulas.

(c) $I_x = \iint_D y^2(x^2+y^2)\,dA$

$= \int_0^{\pi/2} \sin^2\theta\,d\theta \cdot \int_0^1 r^5\,dr$

$\vdots$

(d) $I_y = \iint_D x^2(x^2+y^2)\,dA$

(e) $I_0 = I_x + I_y$ (the moment about the origin)

The moment of inertia is to rotation what mass is to linear motion.

We will skip the radius of gyration.

Probability

The joint density fct of the random variables X & Y is a fct f of two variables s.t. the probability that (X, Y) lies in D is:

$$Pr((X, Y) \in D) = \iint_D f(x,y)\, dA.$$

Note: $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$ ... why?

ex3: Suppose X & Y are random variables w/ joint density fct.

$$f(X,Y) = \begin{cases} 0.1\, e^{-(0.5X + 0.2Y)} & , \quad X, Y \geq 0 \\ 0 & , \quad \text{else} \end{cases}$$

a) verify f is a joint density fct.

b) find $Pr(X \leq 2 \text{ and } Y \leq 4)$

c) find $Pr(Y \geq 1)$

d) Find the expected value of X:

$$\mu_1 = \iint_{\mathbb{R}^2} x \cdot f(x,y)\, dA.$$

(a) $$\iint_{\mathbb{R}^2} f(X, Y)\, dA = \int_0^\infty \int_0^\infty 0.1 e^{-(0.5X + 0.2Y)}\, dX\, dY$$

$$= 0.1 \int_0^\infty e^{-0.5X}\, dX \int_0^\infty e^{-0.2Y}\, dY$$

$$= 0.1 \left[-2e^{-0.5X}\right]_0^\infty \left[-5e^{-0.2Y}\right]_0^\infty$$

$$= 1$$

(b) $\Pr(X \leq 2 \text{ and } Y \leq 4)$

$$= 0.1 \left[-2e^{-0.5X}\right]_0^2 \left[-5e^{-0.2Y}\right]_0^4$$

$$= (e^{-1} - 1)(e^{-.8} - 1)$$

$$= 0.348$$

(c) $\Pr(Y \geq 1) = 1 - \Pr(Y \leq 1)$

$$= 1 - \left[e^{-0.5X}\right]_0^\infty \left[e^{-0.2Y}\right]_0^1$$

$$= 1 - (0 - 1)(e^{-.2} - 1)$$

$$= 0.819$$

(d) Find the expected value of X:

$$\mu_1 = \iint_{\mathbb{R}^2} X\, f(X, Y)\, dA$$

$$= \int_0^\infty \int_0^\infty X \cdot 0.1\, e^{-(0.5X + 0.2Y)}\, dA$$

$$= +0.1 \left[-\frac{1 + 0.5X}{0.5^2} e^{-0.5X} \right]_0^\infty \left[-5\, e^{-0.2Y} \right]_0^\infty$$

$$= 0.1 \frac{1}{0.5^2} \cdot 5$$

$$= 2$$

Other notes:

Formulas for the second moments.

moment of inertia about:

the x-axis: $I_x = \iint_D y^2 \rho(x,y)\, dA$

the y-axis: $I_y = \iint_D x^2 \rho(x,y)\, dA$

the pole: $I_0 = I_x + I_y$

The radius of gyration WRT:

the x-axis: $\bar{\bar{y}} = \frac{I_x}{m}$

the y-axis: $\bar{\bar{x}} = \frac{I_y}{m}$

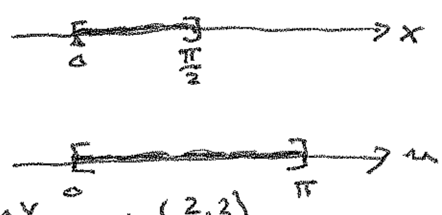
$(\bar{\bar{x}}, \bar{\bar{y}})$ is the pt where the mass could be concentrated w/o changing the moment of inertia.

Scaling Factors

$$\int_0^{\pi/2} \sin(2x)\, dx = \int_0^{\pi} \sin u \cdot \frac{1}{2}\, du = 1$$

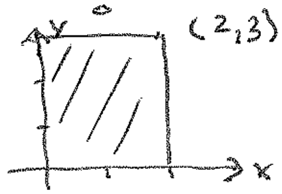
Let $u = 2x dx$
$du = 2dx$

scaling factor.

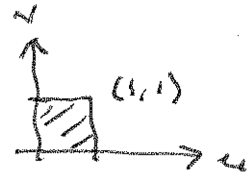


$$\int_0^{\pi/2} \int_0^{\pi/3} \sin(2x) \sin(3y)\, dy\, dx$$

Let $u = 2x$, $v = 3y$
$du = 2dx$, $v = 3dy$



$$= \int_0^{\pi} \int_0^{\pi} \sin(u) \sin(v) \frac{1}{2} \cdot \frac{1}{3}\, du\, dv$$



$= 2 \cdot 2 \cdot \frac{1}{6} = \frac{2}{3}$

$\boxed{\begin{matrix}15.4 \\ 10/10\end{matrix}}$

In 15.9, we will learn about using the Jacobian for change of variables.

Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

polar to rectangular.