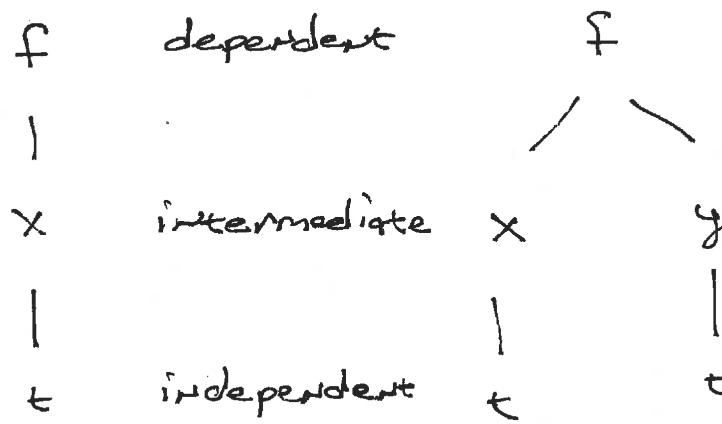
14.5 : Chain Rule

recall, if $w(t) = f(x(t))$, then

$$w'(t) = f'(x(t)) \cdot x'(t).$$

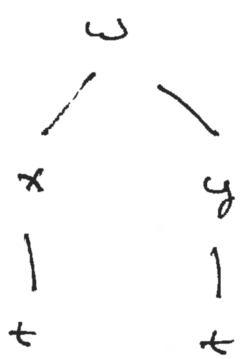
what do we do when $w(t) = f(x(t), y(t))$?



$$\frac{df}{dx} \cdot \frac{dx}{dt} \qquad \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

<u>Ex 1</u>: If $w = 3x^2 + 2xy - y^2$

where $x = \cos t$ & $y = \sin t$, find $\frac{dw}{dt}$.



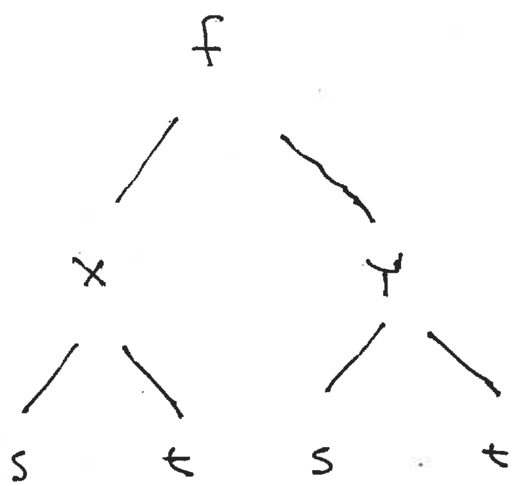
$$\frac{dw}{dt} = (6x + 2y)(-\sin t) + (2x - 2y)\cos t$$

$$= -6\cos t \sin t - 2\sin^2 t + 2\cos^2 t - 2\cos t \sin t$$

$$= -8\cos t \sin t - 2\sin^2 t + 2\cos^2 t$$

you can check by subbing 1st.

If $f(\overset{x}{\overset{\shortparallel}{g(s,t)}}, \overset{y}{\overset{\shortparallel}{h(s,t)}})$, how do we find f_s & f_t.



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

ex2: If $z = \cot\left(\frac{v}{u}\right)$ & $u = 2s - 3t$
$v = 5s + t$

Find $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s}$$

$$= -\csc^2\left(\frac{v}{u}\right)\cdot\frac{-v}{u^2}\cdot 2 + -\csc^2\left(\frac{v}{u}\right)\cdot\frac{1}{u}\cdot 5$$

$$= \frac{2(5s+t)}{2s-3t}\csc^2\left(\frac{5s+t}{2s-3t}\right) - \frac{5}{2s-3t}\csc^2\left(\frac{5s+t}{2s-3t}\right)$$

Find $\frac{\partial z}{\partial t}$ as a HW exercise.

Down the road, we will be learning about cylindrical coords which will require that we rewrite x & y in terms of r & θ

Ex 3: If $z = f(x,y) = x^2 + y^2 + 1$ where $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$, and $\frac{\partial^2 z}{\partial\theta\,\partial r}$

a) $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= 2x\cdot\cos\theta + 2y\sin\theta \quad *$$

stop here until after (c)

$$= 2r\cos^2\theta + 2r\sin^2\theta$$

b) $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= 2x(-r\sin\theta) + 2y(r\cos\theta)$$

stop here until after (c)

$$= -2r^2\sin\theta\cos\theta + 2r^2\sin\theta\cos\theta$$

$$= 0$$

c) $$\frac{\partial^2 z}{\partial\theta\,\partial r} = \frac{\partial^2 z}{\partial x\,\partial r}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y\,\partial r}\frac{\partial y}{\partial \theta}$$

$$= 2\cos\theta\cdot(-r\sin\theta) + 2\sin\theta(r\cos\theta)$$

$$= 0$$

note: by Clairaut's Thm

$\frac{\partial^2}{\partial r\,\partial\theta} = 0$ as well... which is obvious from (b.).

Using the chain rule to perform implicit differentiation.

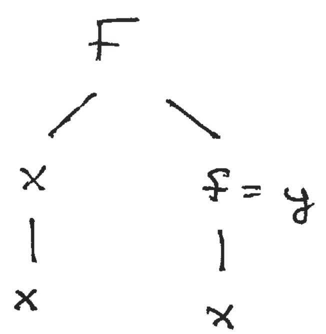
Suppose $F(x,y)=0$ defines y implicitly as a fct of x. (begin ex 4).

That is $y=f(x)$ where $F(x,f(x))=0 \ \forall x \in D_f$.

If f is differentiable, then by the chain rule:

$$0 = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$

solve for $\frac{dy}{dx}$



$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Ex 4: Find $\frac{dy}{dx}$ if $y^5+x^2y^3 = 1+ye^{x^2}$

$$\Rightarrow 0 = 1 - y^5 - x^2y^3 + ye^{x^2} = F(x,y)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{-2xy^3 + 2xye^{x^2}}{-5y^4 - 3x^2y^2 + e^{x^2}}$$