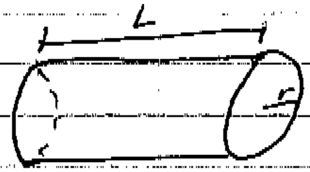
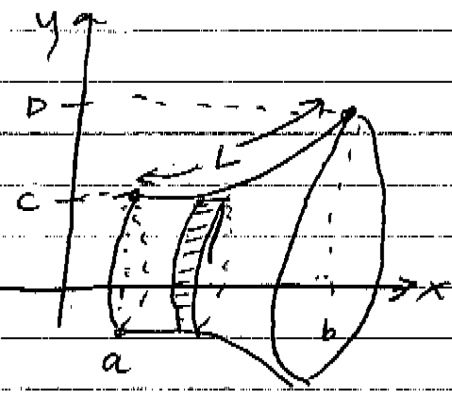
Emma Miller
5/27/09
CALC 125

8.2 - Surface Area of Revolution



cylinder
(no ends)
$SA = 2\pi r L$



$r = y$ value

$$L = \underbrace{\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}_{\text{length of diff. element}}$$

If we want SA, we need length and circum.

a fcn of x

$$SA = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

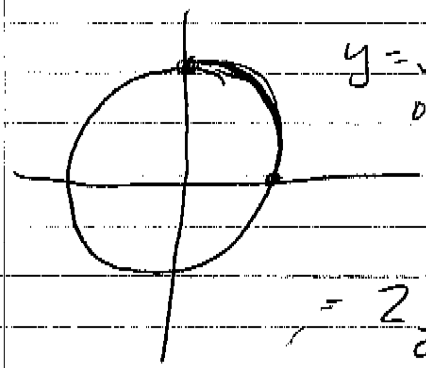
or write as:

$$SA = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

↳ gives SA formed by rotating $y = f(x)$ about x-axis on $a \le x \le b$

8.2

Example 1: Derive SA of sphere w/ radius r



$y = \sqrt{r^2 - x^2}$ on $[0, r]$ $\qquad y' = \frac{-x}{\sqrt{r^2 - x^2}}$

$$SA = 2\int_0^r 2\pi\sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx$$

$$= 2\int_0^r 2\pi\sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}}$$

$$= 4\pi r \int_0^r dx$$

$$= 4\pi r [x]_0^r = 4\pi r^2$$

What if we're given x in terms of y ($x = g(y)$)

$$SA = \int_c^d 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$ → SA of surface formed by rotating $x = g(y)$ about the x axis $c \le y \le d$

Ex. 1 Revisited:

$x = \sqrt{r^2 - y^2} \qquad x' = \frac{-y}{\sqrt{r^2 - y^2}}$

$$SA = 2\int_0^r 2\pi y\sqrt{1 + \frac{y^2}{r^2 - y^2}}\,dy$$

$$= \cancel{2}\, 4\pi\int_0^r y\sqrt{\frac{r^2}{r^2 - y^2}}\,dy$$

$$= 4\pi r\int_0^r \frac{y}{\sqrt{r^2 - y^2}}\,dy$$ let $u = r^2 - y^2$ $\quad du = -2y\,dy$

$$4\pi r\int_{r^2}^0 -\frac{1}{2}\cdot\frac{1}{\sqrt{u}}\,du$$

$$= -4\pi r\left[u^{1/2}\right]_{r^2}^0 = -4\pi r\left(0 - \sqrt{r^2}\right) = 4\pi r^2$$

Example 2: Rotate $x=g(y)$ about y-axis on $c \le y \le d$, what's SA?

$$SA = \int_c^d 2\pi g(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

OR

x is fcn

$$= \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

OR if you have $y=f(x)$ but rotating about y-axis $(a \le x \le b)$

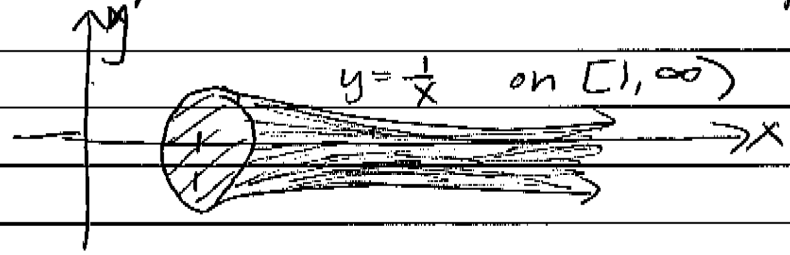
x is variable

$$SA = \int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Notetaker's name Emma Miller Class CALC 125 Date 5/29/09

8.2 - Surface Area

Example 3: Gabriel's Trumpet

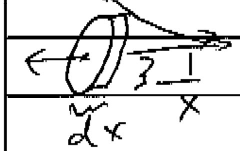


$y = \frac{1}{x}$ on $[1, \infty)$

$y = \frac{1}{x}$

$\frac{dy}{dx} = -\frac{1}{x^2}$

A) What is the volume of Gabriel's Trumpet?



$V = \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx$ ← improper integral

$V = \lim_{t \to \infty} \int_1^t \frac{\pi}{x^2} dx$ ← converges by p-test

$V = \lim_{t \to \infty} \left[\frac{-\pi}{x}\right]_1^t$

$V = \lim_{t \to \infty} \left(\cancel{\frac{-\pi}{t}} + \pi\right) = \pi$

B) What's SA of Gabriel's Trumpet

$SA = \int_1^{\infty} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_1^{\infty} \frac{2\pi}{x} \sqrt{1 + \frac{1}{x^4}}\, dx$

$= \int_1^{\infty} \frac{2\pi}{x} \sqrt{\frac{x^4 + 1}{x^4}}\, dx$

$= \int_1^{\infty} \frac{2\pi}{x^3} \sqrt{x^4 + 1}\, dx$

substitution

let $u = x^2$

$\frac{du}{2} = x\,dx$

$\int_1^{\infty} \frac{2\pi}{u^2} \sqrt{u^2 + 1}\, du$

Notetaker's name Emma Miller Class CALC 125 Date 5/29/09

8.2

$\frac{1}{\cancel{2}}\int_1^\infty \frac{\cancel{2}\pi\sqrt{u^2+1}}{u^2}\,du$

TRIG sub

go back to limits later $\Rightarrow * \int_1^\infty \pi\frac{1}{u^2}\sqrt{u^2+1}\,du$ $u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$\pi\int\frac{1}{\tan^2\theta}\sqrt{\tan^2\theta+1}\cdot\sec^2\theta\,d\theta$

$\pi\int\frac{\sec^3\theta\,d\theta}{\tan^2\theta} = \pi\int\frac{d\theta}{\cos\theta\sin^2\theta}\cdot\frac{\cos\theta}{\cos\theta}$

sub 2

$= \pi\int\frac{\cos\theta\,d\theta}{\cos^2\theta\sin^2\theta} = \pi\int\frac{\cos\theta\,d\theta}{(1-\sin^2\theta)\sin^2\theta}$ $w = \sin\theta$

$dw = \cos\theta\,d\theta$

partial fractions $\Rightarrow = \pi\int\frac{dw}{(1-w^2)w^2} = \frac{A}{1-w}+\frac{B}{1+w}+\frac{C}{w}+\frac{D}{w^2}$

$1 = A(1+w)w^2 + B(1-w)w^2 + C(1-w^2)w + D(1-w^2)$

$w=0: 1 = 1D;\ D=1$ $C=0$

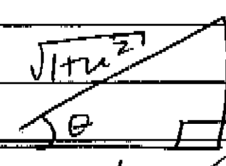
$w=1: 1 = 2A;\ A = 1/2$ $\pi\int\frac{dw}{(1-w^2)\cancel{w}} = \frac{1/2}{(1+w)} + \frac{1/2}{w} + \frac{1}{w^2}$

$w=-1: 1 = 2B;\ B = 1/2$

$\pi\left(\frac{-1}{2}\ln|1-w| + \frac{1}{2}\ln|1+w| + \left(-\frac{1}{w}\right)\right) + C$

$\pi\left(\frac{1}{2}\ln\left|\frac{1+w}{1-w}\right| - \frac{1}{w}\right) + C$ ← sub back

$\pi\left(\frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \frac{1}{\sin\theta}\right) + C$



sub back

$\pi\left(\frac{1}{2}\ln\left|\frac{1+\frac{u}{\sqrt{1+u^2}}}{1-\frac{u}{\sqrt{1+u^2}}}\right| - \frac{\sqrt{1+u^2}}{u}\right) + C$

$\pi\left(\frac{1}{2}\ln\left|\frac{\sqrt{1+u^2}+u}{\sqrt{1+u^2}-u}\right| - \frac{\sqrt{1+u^2}}{u}\right)$

$\pi\left(\frac{1}{2}\ln\left|\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right| - \frac{\sqrt{1+x^4}}{x^2}\right) + C$

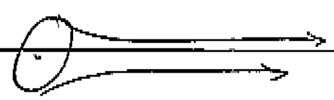
Notetaker's name Emma Miller Class MAC 125 Date 5/29/09

8.2

$$\pi \lim_{t\to\infty} \left[\frac{1}{2}\ln\left|\frac{\sqrt{x^4+1}+x^2}{\sqrt{1+x^4}-x^2}\right| - \frac{\sqrt{1+x^4}}{x^2}\right]_1^t$$

* Need to figure out what happens @ t, since @ 1, it = a #

Need: $\lim_{t\to\infty}\left(\frac{1}{2}\ln\left|\overbrace{\frac{\sqrt{1+t^4}+t^2}{\sqrt{1+t^4}+t^2}}^{\to\infty}\right| - \overbrace{\frac{\sqrt{1+t^4}}{t^2}}^{\to 1}\right) = \infty$

SO  π = volume

∞ = SA

Ex 3 Revisited

$$\int_1^\infty \frac{2\pi}{x^3}\sqrt{1+x^4}\,dx > \int_1^\infty \frac{2\pi}{x^3}\sqrt{x^4}\,dx$$

$$\Leftrightarrow 2\pi\int_1^\infty \frac{1}{x}\,dx$$

↳ diverges by p-test