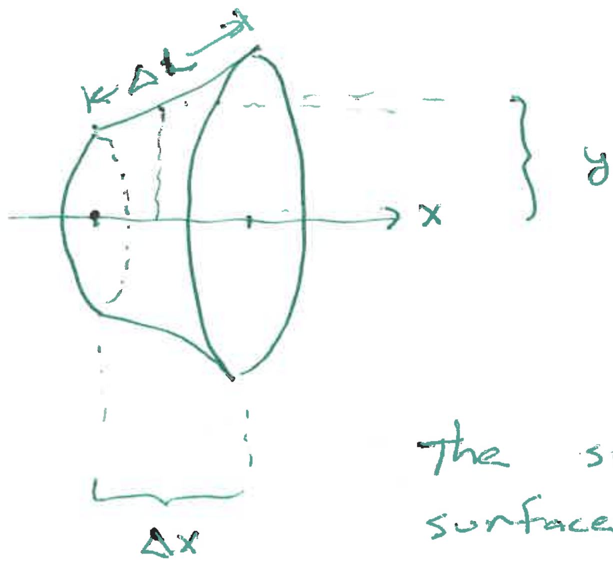
8.2: Area of a surface of revolution.

recall $\Delta L = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$ and

$$dL = \sqrt{1 + (f'(x))^2}\, dx$$

if we rotate about the x-axis



$$\Delta S = 2\pi y \Delta L$$

$$dS = 2\pi y \sqrt{1 + (y')^2}\, dx$$

The surface area of the surface obtained by rotating $y = f(x)$ on $[a, b]$ about the x-axis...

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$$

& if $x = g(y)$ on $[c, d]$ is rotated about the y-axis

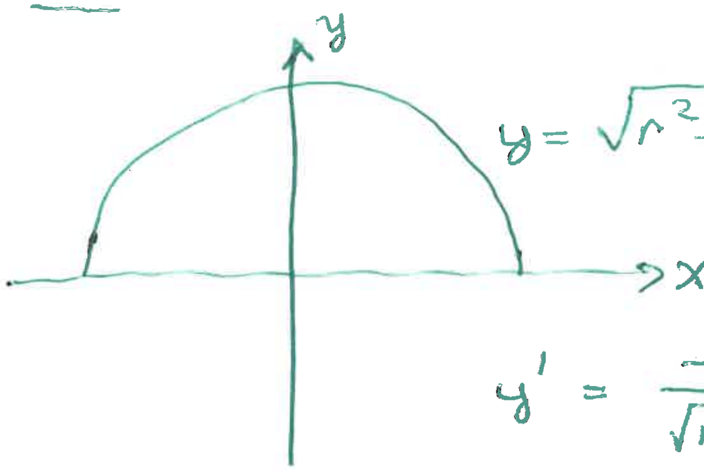
$$S = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy.$$

ex1 Set up an integral to represent the SA of the surface formed by rotating $y = \sqrt[3]{x}$ on $1 \le y \le 2$ about the given axis.

About the x-axis: $\int_1^8 2\pi x \sqrt{1 + \left(\frac{1}{3x^{2/3}}\right)^2}\, dx$

About the y-axis: $\int_1^2 2\pi y^3 \sqrt{1 + (3y^2)^2}\, dy$

ex 2: Verify the SA of a sphere.



$$y = \sqrt{r^2 - x^2}$$

$$y' = \frac{-x}{\sqrt{r^2-x^2}}$$

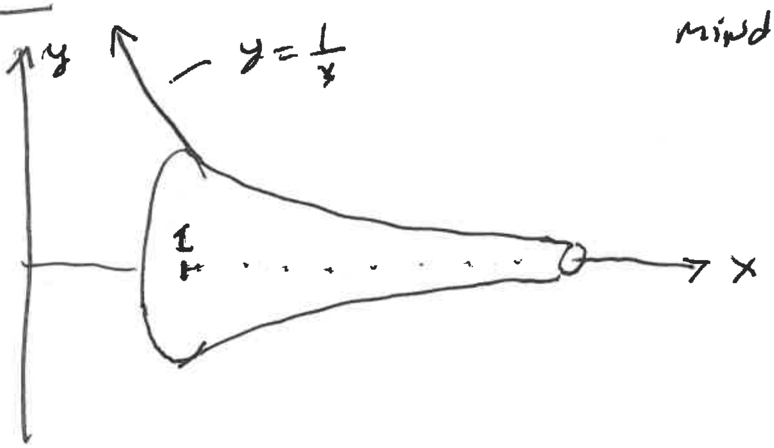
$$SA = \int_{-r}^{r} 2\pi \sqrt{r^2-x^2} \left(1 + \left(\frac{x}{\sqrt{r^2-x^2}} \right)^2 \right)^{1/2} dx$$

$$= 4\pi \int_0^r \sqrt{r^2-x^2} \sqrt{\frac{r^2 - x^2 + x^2}{r^2-x^2}} \, dx$$

$$= 4\pi \int_0^r \cancel{\sqrt{r^2-x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2-x^2}}} \, dx$$

$$= 4\pi r^2.$$

ex3: Gabriel's Trumpet (a classic mind bender)



Gabriel's trumpet is formed by rotating $y=\frac{1}{x}$ on $[1,\infty)$ about the x-axis.

Find the volume (A) and surface Area (B) of Gabriel's Trumpet.

(A) Volume of Gabriel's Trumpet.

$$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \lim_{t\to\infty} \int_1^t \pi x^{-2}\, dx$$

$$= \lim_{t\to\infty} \left[-\pi \frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty} \left(\pi - \frac{\pi}{t}\right)$$

$$= \pi$$

so it will take π "gallons" of gold paint to fill Gabriel's Trumpet.

(B) Surface Area of Gabriel's Trumpet.

$$y = \frac{1}{x} \quad ; \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

$$SA = \int_1^\infty \frac{2\pi}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$$

$$= \int_1^\infty \frac{2\pi}{x^3}\sqrt{x^4+1}\, dx$$

consider

$$* \quad I = \int \frac{2\pi}{x^3}\sqrt{x^4+1}\, dx = \int \frac{2\pi x}{x^4}\sqrt{x^4+1}\, dx$$

$\boxed{\text{Trig sub}}$ Let $x^2 = \tan\theta$

$$2x\,dx = \sec^2\theta\, d\theta$$

$$I = \pi \int \frac{\sqrt{\tan^2\theta + 1}}{\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \pi \int \frac{\sec^3\theta}{\tan^2\theta}\, d\theta$$

$$= \pi \int \frac{d\theta}{\sin^2\theta\cos\theta}$$

$$= \pi \int \frac{\cos\theta\, d\theta}{\sin^2(1-\sin^2\theta)}$$

Let $u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= \pi \int \frac{du}{u^2(1-u^2)}$$

$\boxed{\text{partial fractions}}$

$$\frac{1}{u^2(1-u^2)} = \frac{A}{1+u} + \frac{B}{1-u} + \frac{C}{u} + \frac{D}{u^2}$$

$$\Rightarrow \quad 1 = A(1-u)u^2 + B(1+u)u^2 + Cu(1-u^2) + D(1-u^2)$$

$u=0$: $D=1$

$u=1$: $2B=1 \Rightarrow B = 1/2$

$u=-1$: $2A=1 \Rightarrow A = 1/2$

match u^1 coefficients: $C=0$.

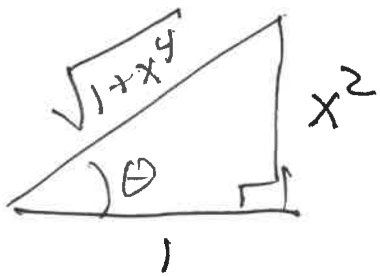
$$I = \pi \int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u} + \frac{1}{u^2} \right) du$$

$$= \frac{\cancel{2}\pi}{2}\left(\ln(1+u) - \ln(1-u) - \frac{2}{u} \right) + C$$

$$= \frac{\pi}{2}\left(\ln\left|\frac{1+u}{1-u}\right| - \frac{2}{u} \right) + C$$

$\boxed{\text{sub back}}$

$$= \frac{\pi}{2}\left(\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \frac{2}{\sin\theta} \right) + C$$



$$\sin\theta = \frac{x^2}{\sqrt{1+x^4}}$$

$$I = \frac{\pi}{2}\left(\ln\left|\frac{1+\frac{x^2}{\sqrt{1+x^4}}}{1-\frac{x^2}{\sqrt{1+x^4}}}\right| - \frac{2\sqrt{1+x^4}}{x^2}\right) + C$$

$$= \frac{\pi}{2}\left(\ln\left|\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right| - \frac{2\sqrt{1+x^4}}{x^2}\right) + C$$

$$* \quad \text{so} \quad \int_1^\infty \frac{2\pi}{x^3}\sqrt{x^4+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{2\pi}{x^3}\sqrt{x^4+1}\,dx$$

$$= \frac{\pi}{2}\lim_{t\to\infty}\left(\ln\overset{\to\infty}{\left|\frac{\sqrt{1+t^4}+t^2}{\sqrt{1+t^4}-t^2}\right|} - \overset{\to 1}{\frac{2\sqrt{1+t^4}}{t^2}}\right) - \#$$

$$= \infty$$

So the SA of Gabriel's Trumpet is infinite and no amount of gold can cover the horn.

<u>Conclusion</u>: The paradox of Gabriel's Trumpet is that it can be filled, but not coated.

(B) Take Two (the easy way).

$$SA = \int_1^\infty \frac{2\pi}{x^3}\sqrt{1+x^4}\,dx \geq 2\pi\int_1^\infty \frac{\sqrt{x^4}}{x^3}\,dx$$

$$= 2\pi\int_1^\infty \frac{dx}{x} \quad \text{diverges by p-test.}$$

So, the SA is ∞ (diverges) by comparison.