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Section 7.8

Improper Integrals

Ex:1 Find the area under $f(x) = \frac{1}{x^3}$ on $[1, t]$

$$\int_1^t \frac{dx}{x^3} = \left[-\frac{1}{2x^2}\right]_1^t = -\frac{1}{2t^2} + \frac{1}{2}$$

what happens as $t \to \infty$

$$\int_1^\infty \frac{dx}{x^3} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^3} = \lim_{t\to\infty} \left(-\frac{1}{2x^2} + \frac{1}{2}\right) = \frac{1}{2}$$

Example of a convergent improper integral

create a new parameter t to use as the upper limit.

Ex2 $$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} [\ln|x|]_1^t = \lim_{t\to\infty} (\ln|t| - \ln|1|) = \infty$$ ← not a #

Example of a divergent improper integral

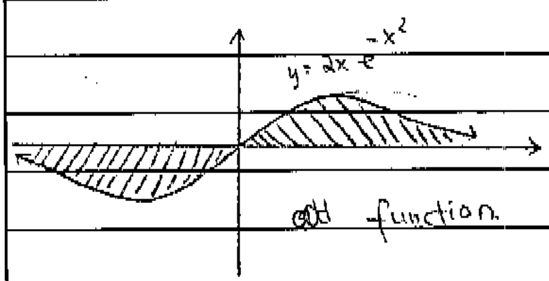
Ex:3 $$\int_{-\infty}^{\infty} 2x e^{-x^2} dx = \int_{-\infty}^{0} 2xe^{-x^2} dx + \int_0^\infty 2xe^{-x^2} dx$$

$$= \lim_{t\to\infty} \int_{-t}^{0} 2xe^{-x^2} dx + \lim_{j\to\infty} \int_0^j 2xe^{-x^2} dx$$

$$= \lim_{t\to\infty} \left[-e^{-x^2}\right]_{-t}^{0} \quad \Bigg| \quad = \lim_{j\to\infty} \int_0^j 2xe^{-x^2} dx = \lim \left[-e^{-x^2}\right]_0^j$$

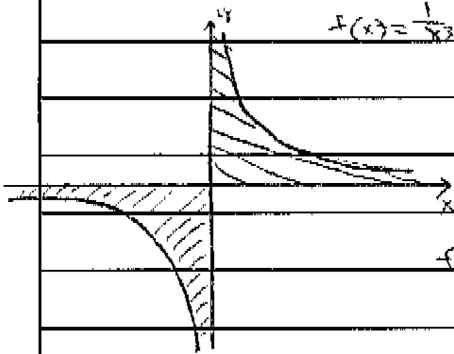
$$= \lim_{t\to\infty} \left(-1 + \cancel{e^{-t^2}}\right) = -1 \quad \Bigg| \quad = \lim_{j\to\infty} \left(\cancel{-e^{-j^2}} + 1\right) = 1$$

$-1 + 1 = 0$

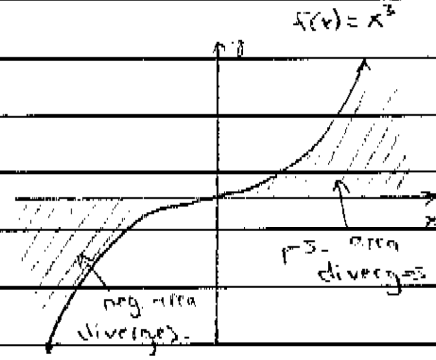


odd function

fails in case 1.



function is not continuous.



$\int_{-\infty}^{\infty} f(x)\,dx = 0$

is $f(x)$ is odd.

Ex:4 $\displaystyle\int \frac{dx}{\underbrace{x^p}_{x^{-p}}} = \begin{cases} \ln|x|, & p=1 \\ \frac{x^{1-p}}{1-p} & p\neq 1 \end{cases}$

now $\displaystyle\int_1^{\infty} \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left[\begin{cases} \overset{\text{case 1}}{\ln x}, & p=1 \\ \overset{\text{case 2}}{\frac{x^{1-p}}{1-p}} & p\neq 1\end{cases}\right]_1^t$

Case 1 $p=1$

$\displaystyle\lim_{t\to\infty}(\ln t - \ln 1) = \infty$

diverges

case 2 $p\neq 1$

$\displaystyle\lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \underbrace{\frac{1^{1-p}}{1-p}}_{\frac{1}{1-p}}\right)$

depends on p

$p>1$ converge

$p<1$ diverge

Conclusion from Ex. 4.

$\displaystyle\int_1^{\infty}\frac{dx}{x^p}$ diverges if $p\leq 1$

converges is $p>1$

this is called the p- test.

Ex:5 (A) $\displaystyle\int_1^{\infty}\frac{dx}{x^{1/3}}$ diverges (B) $\displaystyle\int_1^{\infty}\frac{dx}{x}$ diverges (C) $\displaystyle\int_1^{\infty}\frac{dx}{x^2}$ converges.

Question: what about other fixed limits

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$$\int_2^\infty \frac{dx}{x^2} = \int_1^\infty \frac{dx}{x^2} - \int_1^2 \frac{dx}{x^2}.$$

Ex:5 $$\int_0^\infty \frac{dx}{x^2} = \underbrace{\underbrace{\int_0^1 \frac{dx}{x^2}}_{\text{undef. } \frac{1}{0}}}_{\text{improper int.}} + \underbrace{\int_1^\infty \frac{dx}{x^2}}_{\text{converges}}$$

$$\int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \left(-\frac{1}{x}\right]_t^1 = \lim_{t\to 0} \left(-1 + \frac{1}{t}\right) = \infty$$

$\int_0^\infty \frac{dx}{x^2}$ diverges.

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(Improper Integrals)

recall from yesterday

Ex: 6

$$\int_0^{\infty} \frac{dx}{x^2} = \underbrace{\int_0^1 \frac{dx}{x^2}}_{\text{diverges}} + \underbrace{\int_1^{\infty} \frac{dx}{x^2}}_{\text{converges by p-test.}}$$

p-test

$$\int_1^{\infty} \frac{dx}{x^p}$$

converges: $p > 1$

diverges: $p \geq 1$

p-test

$$\int_0^1 \frac{dx}{x^p}$$

converges: $p < 1$

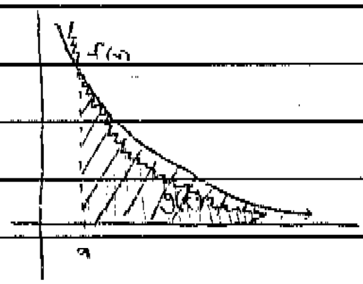
diverges: $p \geq 1$

Ex: 7 $$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t \to 0^+} \left[2\sqrt{x}\right]_t^1 = \lim_{t \to 0^+} \left(2 - 2\overset{0}{\sqrt{t}}\right) = 2$$

converges.

Ex: 8 $$\int_0^3 \frac{dx}{\sqrt{x}} = \underbrace{\int_0^1 \frac{dx}{\sqrt{x}}}_{\text{convergent by p-test.}} + \int_1^3 \frac{dx}{\sqrt{x}}$$

Comparison test for improper integrals.



Suppose $\int_a^{\infty} f(x)\,dx$ converges

then $\int_a^{\infty} g(x)\,dx$ converges.

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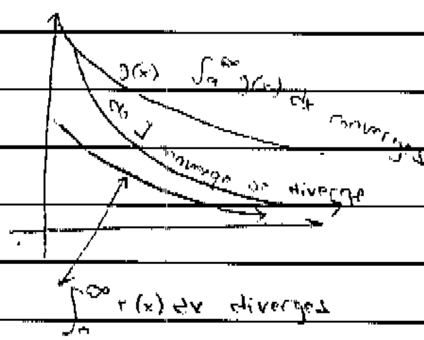
Suppose $\int_a^\infty f(x)\,dx$ diverges

then $\int_a^\infty h(x)\,dx$ also diverges

The concept

Some function, and you want to know if it converges or diverges. $\int_a^\infty f(x)\,dx$



1. find func. above
2. find func. below.

Ex:9 $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx \le \int_1^\infty \frac{dx}{x^2}$ converges by p-test.

hint making bigger.

$\frac{x}{\sqrt{1+x^6}} \le \frac{x}{\sqrt{x^6}} = \frac{1}{x^2}$

Hence, $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ converges by the comparison test.

Ex:10

$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx.$ biggest numerator is 1.

So, $\frac{e^{-x}}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ on $[0,1]$

$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx \le \int_0^1 \frac{dx}{\sqrt{x}}$ convergest by p-test.

Hence $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ converges by the comparison test.

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Ex: 11 $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx \leq \int_1^\infty \frac{2+e^{-1}}{x}\,dx = 2+\frac{1}{e}\int_1^\infty \frac{dx}{x}$ diverges by p-test.

$\displaystyle\frac{2+0}{x} \leq \frac{2+e^{-x}}{x} \leq \frac{2+e^{-1}}{x}$

$\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx \geq \int_1^\infty \frac{dx}{x}$ diverges by p-test

Hence $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ diverges by comparison.