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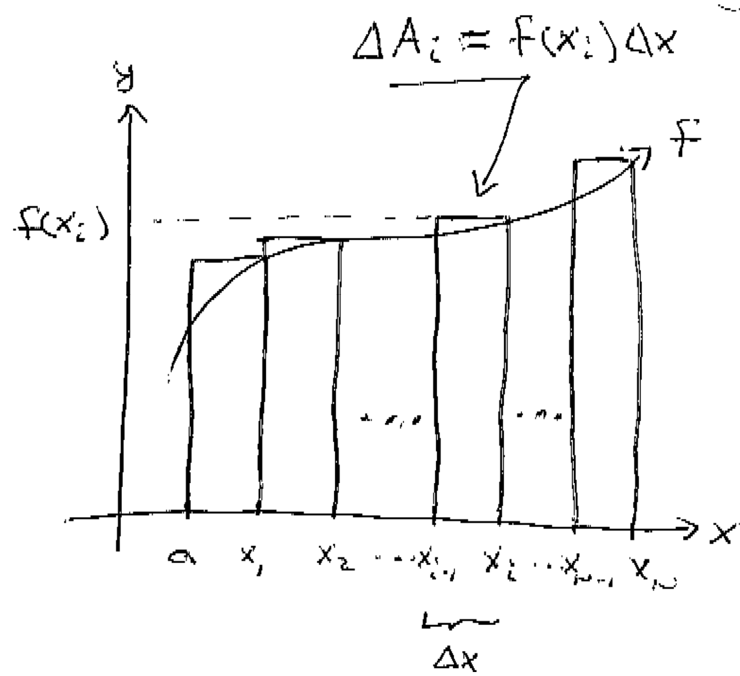
Generalizing from the previous section and using sigma notation;

To find the area under $y = f(x)$ from $x = a$ to $x = b$ using n rectangles of equal width.



Let $\Delta x = \dfrac{b-a}{n}$

$$a = x_0$$

$$b = x_n$$

so $x_1 = a + \Delta x$

$$x_2 = a + 2\Delta x$$

$$\vdots$$

$$x_i = a + i\Delta x$$

Right endpoints

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_i)\Delta x + \dots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$ (approx area) w/a Riemann sum.

and the exact area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Left endpoints

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{i-1})\Delta x + \dots + f(x_{n-1})\Delta x$$

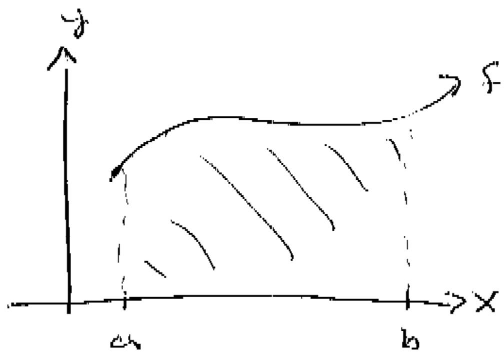
$$= \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

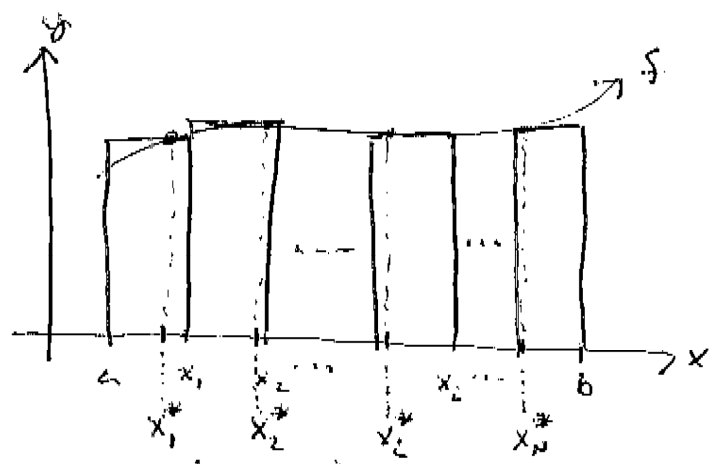
$$= \sum_{i=0}^{n-1} f(x_i)\Delta x$$ (the approx area w/a Riemann sum).

and the exact area is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

To find the exact area, find the limit of the Riemann sums. This can be stated ~~gen~~ more generally





here we take our sample points arbitrarily from each interval. That is $x_0 \le x_1^* \le x_1$; $x_1 \le x_2^* \le x_2$; ... ; $x_{i-1} \le x_i^* \le x_i$

so, we can find the exact area under the fct w/

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{where } x_{i-1} \le x_i^* \le x_i$$

$\boxed{\begin{matrix} 5.2 \\ 2/5 \end{matrix}}$

This allows us to state the following definition;

Defn! The Definite Integral

If f is a cont. fct on $a \le x \le b$, $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ for $i = 0, 1, 2, \ldots, n$, and $x_{i-1} \le x_i^* \le x_i$. Then the definite integral of f from a to b is!

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

~~Notation~~ NOTE: While the derivative is a fct, the definite integral is a number and does not depend upon x.

NOTATION: $\int$ integral sign $f(x)$ integrand

a, b are the lower & upper limits of integration dx integrate w/respect to this variable.

NOTE since $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is a Riemann sum, we have defined the definite integral as the limit of Riemann sums.

ex1: Express $\int_2^4 x^3\,dx$ as the limit of Riemann sums.

ex2: express $\lim_{n\to\infty} \sum_{i=1}^{n} \left[(1+\frac{3i}{n}) - (1+\frac{3i}{n})^2 \ln(1+\frac{3i}{n})\right]\frac{3}{n}$ as an integral $[1, 4]$

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Good stuff to know

a) $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

b) $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

c) $\sum_{i=1}^{N} i^3 = \left[\frac{N(N+1)}{2}\right]^2$

d) $\sum_{i=1}^{N} c = Nc$

e) $\sum_{i=1}^{N} c \cdot a_i = c \sum_{i=1}^{N} a_i$

f) $\sum_{i=1}^{N} (a_i \pm b_i) = \sum_{i=1}^{N} a_i \pm \sum_{i=1}^{N} b_i$

Ex 3: Evaluate the Riemann sum for $f(x) = x^2 - 3x$ taking sample points to be left end points and $a = 1$, $b = 5$, and $N = 4$.

Ex 4: Evaluate $\int_0^5 (x^2 - 3x)\,dx$

Ex 5: Set up an expression for $\int_0^{\pi} \sin(x)\,dx$ as a limit of sums.

Evaluate the following geometrically.

Ex 6: $\int_{-1}^{3} (3 - 2x)\,dx$

Ex 7: $\int_{-3}^{0} (1 + \sqrt{9 - x^2})\,dx$

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The midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x \quad \text{where } \Delta x = \frac{b-a}{n}$$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. ← the midpoint on $[x_{i-1}, x_i]$.

Ex 8: Estimate $\int_0^\pi \sin(x)\,dx$ using a Riemann sum w/ 3 subintervals. Compare left, right, and midpoint approximations.

Properties of the Definite Integral

I) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ $\quad \Delta x = \frac{b-a}{n} = -\frac{a-b}{n}$

II) $\int_a^a f(x)\,dx = 0$

III) $\int_a^b c\,dx = c(b-a)$

IV) $\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$

V) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Ex 9: Use the properties of integrals to evaluate $\int_0^1 (6-4x^3)\,dx$ (remember $\int_0^1 x^3\,dx$ from Day 1).

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VI) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

p ~~385~~ 374, "not easy to prove."

Ex 10: write $\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$

as a single integral.

comparison properties of the integral

VII) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

VIII) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

IX) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

If time, prove (VIII)

Ex 11: use IX to estimate $\int_1^2 \frac{dx}{x}$.