Name: Van Nguyen
Class: Math 125 (#6361)
Assignment: Problem Plus (Honor Problems)

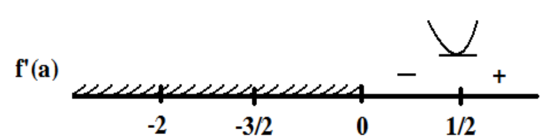
Chapter 5: Problem #2:

Find the minimum value of the area of the region under the curve $y = x + \frac{1}{x}$ from $x = a$ to $x = a + 1.5$, for all a > 0.

- I = $\int_{a}^{a+1.5} \left(x + \frac{1}{x}\right) dx$ = $\left[\frac{1}{2}x^2 + \ln|x|\right]_{a}^{a+1.5}$
  = $\frac{1}{2}(a + \frac{3}{2})^2 + \ln(a + \frac{3}{2}) - \frac{1}{2}a^2 - \ln a$ = $\frac{3}{2}a + \frac{9}{8} + \ln(1 + \frac{3}{2a})$
- Let $f(a) = \frac{3}{2}a + \frac{9}{8} + \ln(1 + \frac{3}{2a})$

- a>0, so domain: (0, ∞)

→ $f'(a) = \frac{3}{2} + \frac{2a(\frac{-3}{2a^2})}{2a+3} = \frac{3}{2} - \frac{3}{a(2a+3)} = \frac{6a^2+9a-6}{2a(2a+3)}$



→ $f'(a) = 0 \leftrightarrow 6a^2 + 9a - 6 = 0$

→ $3(2a^2 + 3a - 2) = 0 \leftrightarrow 3(2a - 1)(a + 2) = 0$

→ $a = \frac{1}{2}$ or $a = -\frac{3}{2}$, but we choose $a = \frac{1}{2}$ since $a > 0$

→ At $a = \frac{3}{2}$, the function has a local minimum.

Therefore, minimum value of the area of the region is:

I = $\frac{3}{2} * \frac{1}{2} + \frac{9}{8} + \ln\left(1 + \frac{3}{2*\frac{1}{2}}\right) = \frac{15}{8} + \ln 4$

$$I = \frac{15}{8} + \ln 4 \approx 3.2612943611199$$

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Chapter 7: Problem #2:

Evaluate $\int \frac{1}{x^7-x} dx$. The straightforward approach would be to start with partial fractions, but that would be brutal. Try a substitution.

- $I = \int \frac{1}{x^7-x} dx = \int \frac{1}{x(x^6-1)} dx$
- Let $u = x^3$ → $u^{1/3} = x$

→ $du = 3x^2 dx \rightarrow \frac{1}{3} * \frac{1}{u^{2/3}} du = dx$

→ $I = \frac{1}{3} \int \frac{1}{u^{1/3} * (u^2-1) * u^{2/3}} du = \frac{1}{3} \int \frac{1}{u * (u^2-1)} du = \frac{1}{3} \int (\frac{A}{u} + \frac{Bu+C}{u^2-1}) du$

- We have $\frac{1}{u*(u^2-1)} = \frac{A}{u} + \frac{Bu+C}{u^2-1}$

→ $A(u^2-1) + (Bu+C)u = 1$

→ $(A+B)u^2 + Cu - A = 1$

→ $\begin{cases} A+B=0 \\ C=0 \\ -A=1 \end{cases} \rightarrow \begin{cases} A=-1 \\ B=1 \\ C=0 \end{cases}$

→ $I = \frac{1}{3} \int (\frac{-1}{u} + \frac{u}{u^2-1}) du = -\frac{1}{3} \ln|x^3| + \frac{1}{6} \ln|x^6-1| + C = \frac{1}{3} \ln|\frac{\sqrt{x^6-1}}{x^3}| + C$

$$\boxed{I = \frac{1}{3} \ln|\frac{\sqrt{x^6-1}}{x^3}| + C}$$

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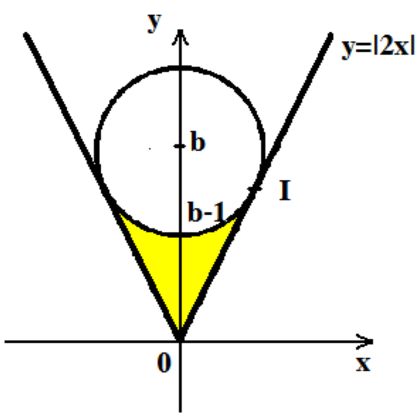
Chapter 7: Problem #13:

The circle with radius 1 shown in the figure touches the curve $y = |2x|$ twice. Find the area of the region that lies between the two curves.



- We have $x^2 + (y-b)^2 = 1$, $(b > 0)$ and $y = |2x|$
- Let's consider the area on the right side between $x^2 + (y-b)^2 = 1$ and $y = |2x| = 2x$ since $x > 0$.
- Intersection: $x^2 + (2x-b)^2 = 1$ (by substitution)

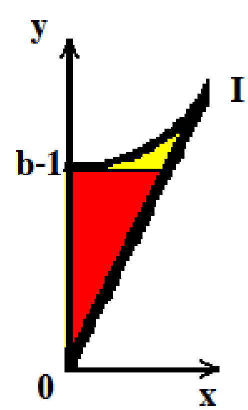
$\rightarrow 5x^2 - 4bx + b^2 - 1 = 0$

- We have: $\Delta = B^2 - 4AC$

$= (-4b)^2 - 4 * 5 * (b^2 - 1) = -4b^2 + 20$

- If $y = 2x$ touches $x^2 + (y-b)^2 = 1$ at 1 point, $\Delta = 0$

  $\rightarrow -4b^2 + 20 = 0 \rightarrow b = -\sqrt{5}$ or $b = \sqrt{5}$



- Choose $b = \sqrt{5}$ since $b > 0 \rightarrow x^2 + (y - \sqrt{5})^2 = 1$
- Intercept: $x = \frac{2}{\sqrt{5}}$, $y = \frac{4}{\sqrt{5}} \rightarrow I(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}})$
- Total Area $= 2 * (Area\ of\ red\ triangle + \ yellow\ area)$
- Total Area $= 2 * \left(\frac{1}{2}(\sqrt{5}-1) * \frac{(\sqrt{5}-1)}{2} + H\right) = 2\left(\frac{(\sqrt{5}-1)^2}{4} + H\right)$
- $H = \int_{\sqrt{5}-1}^{4/\sqrt{5}} \left(\frac{1}{2}y - \sqrt{1-(y-\sqrt{5})^2}\right) dy$
- Let $y - \sqrt{5} = \sin\theta \rightarrow dy = \cos\theta * d\theta$
- $H = \int_{-\pi/2}^{\sin^{-1}(-1/\sqrt{5})} (\frac{1}{2}(\sin\theta + \sqrt{5}) - \cos\theta) \cos\theta * d\theta$

  $= \int_{-\pi/2}^{\sin^{-1}(-1/\sqrt{5})} (\frac{1}{4}\sin(2\theta) + \frac{\sqrt{5}}{2}\cos\theta - \frac{1}{2}\cos(2\theta) - \frac{1}{2}) d\theta$

  $= [-\frac{1}{8}\cos(2\theta) + \frac{\sqrt{5}}{2}\sin\theta - \frac{1}{4}\sin(2\theta) - \frac{1}{2}\theta]_{-\pi/2}^{\sin^{-1}(-1/\sqrt{5})}$

  $= [-\frac{1}{8} + \frac{1}{4}(\sin\theta)^2 + \frac{\sqrt{5}}{2}\sin\theta - \frac{1}{2}\sin\theta\cos\theta - \frac{1}{2}\theta]_{-\pi/2}^{\sin^{-1}(-1/\sqrt{5})}$

$= -\frac{1}{8} + \frac{1}{20} - \frac{1}{2} + \frac{1}{5} - \frac{1}{2}\sin^{-1}(-1/\sqrt{5}) + \frac{1}{8} - \frac{1}{4} + \frac{\sqrt{5}}{2} - \frac{\pi}{4}$

$= \frac{\sqrt{5}-1}{2} - \frac{\pi}{4} - \frac{1}{2}\sin^{-1}(-1/\sqrt{5})$

- Total Area $= 2\left(\frac{(\sqrt{5}-1)^2}{4} + H\right) = 2\left[\frac{(\sqrt{5}-1)^2}{4} + \frac{\sqrt{5}-1}{2} - \frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right]$

$= \sin^{-1}(\frac{1}{\sqrt{5}}) - \frac{\pi}{2} + 2 \approx 0.8928512822$

$$\boxed{Total\ Area = \sin^{-1}(\frac{1}{\sqrt{5}}) - \frac{\pi}{2} + 2 \approx 0.8928512822}$$

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Chapter 7: Problem #11:

If 0< a < b, find $\lim_{t\to 0}\{ \int_0^1 [bx + a(1-x)]^t dx \}^{1/t}$

- Let $I = \int_0^1 [bx + a(1-x)]^t dx = \frac{1}{b-a}\int_0^1 (b-a)[bx + a - ax]^t dx$

  = $\frac{1}{b-a}\int_0^1 (b-a)[(b-a)x + a]^t dx$

  $\rightarrow I = \left[\frac{[(b-a)x+a]^{t+1}}{(b-a)(t+1)}\right]_0^1 = \frac{(b-a+a)^{t+1}-a^{t+1}}{(b-a)(t+1)} = \frac{b^{t+1}-a^{t+1}}{(b-a)(t+1)}$

- $\lim_{t\to 0}\{ \int_0^1 [bx + a(1-x)]^t dx \}^{1/t} = \lim_{t\to 0}\left(\frac{b^{t+1}-a^{t+1}}{(b-a)(t+1)}\right)^{1/t}$

  (= 1^{∞}: $an\ indeterminate\ power$)

  = $\lim_{t\to 0}\left(\frac{b^{t+1}-a^{t+1}}{(b-a)(t+1)}\right)^{1/t} = \lim_{t\to 0} e^{\frac{\ln\left(\frac{b^{t+1}-a^{t+1}}{(b-a)(t+1)}\right)}{t}}$ (*)

  $\rightarrow$ By L'H, we take derivative of the numerator and the denominator of the power of *e* with respect to *t*. Derivative of the denominator is 1, so we just care about the numerator.

  $\rightarrow$We have:

$$\left(\ln\left(\frac{b^{t+1}-a^{t+1}}{(b-a)(t+1)}\right)\right)' = \frac{(b-a)(t+1)}{b^{t+1}-a^{t+1}} * \frac{[b^{t+1}*\ln(b)-a^{t+1}*\ln(a)](b-a)(t+1)-[b^{t+1}-a^{t+1}](b-a)}{(b-a)^2(t+1)^2}$$

$$= \frac{[b^{t+1}*\ln(b)-a^{t+1}*\ln(a)](t+1)-[b^{t+1}-a^{t+1}]}{[b^{t+1}-a^{t+1}](t+1)} = \frac{[b^{t+1}*\ln(b)-a^{t+1}*\ln(a)]}{[b^{t+1}-a^{t+1}]} - \frac{1}{t+1}$$

From (*), we have $\lim_{t\to 0} e^{\frac{\ln\left(\frac{b^{t+1}-a^{t+1}}{(b-a)(t+1)}\right)}{t}} \left(\overline{\overline{byL'H}}\right) e^{\lim_{t\to 0}\{\frac{[b^{t+1}*\ln(b)-a^{t+1}*\ln(a)]}{[b^{t+1}-a^{t+1}]} - \frac{1}{t+1}\}}$

$$= e^{\lim_{t\to 0}\left\{\frac{[b^{t+1}*\ln(b)-a^{t+1}*\ln(a)]}{[b^{t+1}-a^{t+1}]}\right\} + \lim_{t\to 0}\frac{-1}{t+1}} = e^{\lim_{t\to 0}\{\frac{[b^{t+1}*\ln(b)-a^{t+1}*\ln(a)]}{[b^{t+1}-a^{t+1}]}\}} * e^{\lim_{t\to 0}\frac{-1}{t+1}}$$

$$= e^{\frac{b*\ln(b)-a*\ln(a)}{b-a}} * e^{-1} = e^{\frac{\ln(b^b)-\ln(a^a)}{b-a}} * e^{-1} = e^{\frac{1}{b-a}*\ln\left(\frac{b^b}{a^a}\right)} * e^{-1}$$

$$= \left[\frac{b^b}{a^a}\right]^{1/(b-a)} * e^{-1}$$

Therefore, $\boxed{\lim_{t\to 0}\{ \int_0^1 [bx + a(1-x)]^t dx \}^{1/t} = e^{-1} * \left[\frac{b^b}{a^a}\right]^{1/(b-a)}}$