3.7.6 (Model of an epidemic) In pioneering work in epidemiology, Kermack and McKendrick (1927) proposed the following simple model for the evolution of an epidemic. Suppose that the population can be divided into three classes: $x(t)$ = number of healthy people; $y(t)$ = number of sick people; $z(t)$ = number of dead people. Assume that the total population remains constant in size, except for deaths due to the epidemic. (That is, the epidemic evolves so rapidly that we can ignore the slower changes in the populations due to births, emigration, or deaths by other causes.) p 91

Then the model is

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - \ell y$$
$$\dot{z} = \ell y$$

where k and ℓ are positive constants. The equations are based on two assumptions:

(i) Healthy people get sick at a rate proportional to the product of x and y. This would be true if healthy and sick people encounter each other at a rate proportional to their numbers, and if there were a constant probability that each such encounter would lead to transmission of the disease.

(ii) Sick people die at a constant rate ℓ.

The goal of this exercise is to reduce the model, which is a *third-order system*, to a first-order system that can analyzed by our methods. (In Chapter 6 we will see a simpler analysis.)

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a) Show that $x + y + z = N$, where N is constant.

b) Use the $\dot{x}$ and $\dot{z}$ equation to show that $x(t) = x_0 \exp(-kz(t)/\ell)$, where $x_0 = x(0)$.

c) Show that z satisfies the first-order equation $\dot{z} = \ell\left[N - z - x_0 \exp(-kz/\ell)\right]$.

d) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

by an appropriate rescaling.

e) Show that $a \geq 1$ and $b > 0$.

f) Determine the number of fixed points u^* and classify their stability.

g) Show that the maximum of $\dot{u}(t)$ occurs at the same time as the maximum of both $\dot{z}(t)$ and $y(t)$. (This time is called the ***peak*** of the epidemic, denoted t_{peak}. At this time, there are more sick people and a higher daily death rate than at any other time.)

h) Show that if $b < 1$, then $\dot{u}(t)$ is increasing at $t = 0$ and reaches its maximum at some time $t_{\text{peak}} > 0$. Thus things get worse before they get better. (The term ***epidemic*** is reserved for this case.) Show that $\dot{u}(t)$ eventually decreases to 0.

i) On the other hand, show that $t_{\text{peak}} = 0$ if $b > 1$. (Hence no epidemic occurs if $b > 1$.)

j) The condition $b = 1$ is the *threshold* condition for an epidemic to occur. Can you give a biological interpretation of this condition?

k) Kermack and McKendrick showed that their model gave a good fit to data from the Bombay plague of 1906. How would you improve the model to make it more appropriate for AIDS? Which assumptions need revising?

For an introduction to models of epidemics, see Murray (1989), Chapter 19, or Edelstein–Keshet (1988). Models of AIDS are discussed by Murray (1989) and May and Anderson (1987). An excellent review and commentary on the Kermack–McKendrick papers is given by Anderson (1991).

Nonlinear Dynamics and Chaos
by Steven H. Strogatz
Perseus Books (NY) 1994

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6.5.6 (Epidemic model revisited) In Exercise 3.7.6, you analyzed the Kermack–McKendrick model of an epidemic by reducing it to a certain first-order system. In this problem you'll see how much easier the analysis becomes in the phase plane. As before, let $x(t) \geq 0$ denote the size of the healthy population and $y(t) \geq 0$ denote the size of the sick population. Then the model is

$$\dot{x} = -kxy, \qquad \dot{y} = kxy - \ell y$$

where $k, \ell > 0$. (The equation for $z(t)$, the number of deaths, plays no role in the x, y dynamics so we omit it.)

a) Find and classify all the fixed points.
b) Sketch the nullclines and the vector field.
c) Find a conserved quantity for the system. (Hint: Form a differential equation for dy/dx. Separate the variables and integrate both sides.)
d) Plot the phase portrait. What happens as $t \to \infty$?
e) Let (x_0, y_0) be the initial condition. An *epidemic* is said to occur if $y(t)$ increases initially. Under what condition does an epidemic occur?