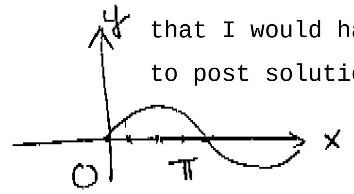
Practice Problems

Thanks to Phuong Tran for working thru these problems for me. If it weren't for her initiative, I don't know that I would have been able to post solutions in a timely manner.
Dust - (March 2009)

2/ $\int_0^{\pi} \sin\theta \, d\theta$.

$n = 4, \ a = 0, \ b = \pi, \ \Delta x = \frac{\pi}{4}$



Leftendpoints.

$$I = \frac{\pi}{4} \cdot \left[f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) \right]$$

$$= \frac{\pi}{4} \left[0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} \right] = \frac{\pi}{4} \left[\sqrt{2} + 1 \right]$$

Rightendpoints:

$$I = \frac{\pi}{4} \left[f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi) \right]$$

$$= \frac{\pi}{4} \left[\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 \right] = \frac{\pi}{4} \left[\sqrt{2} + 1 \right]$$

Trapezoid.

$$I = \frac{\pi}{4 \cdot 2} \left[f(0) + 2f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{3\pi}{4}\right) + f\left(\frac{\pi}{4}\right) \right]$$

$$= \frac{\pi}{8} \left[0 + \sqrt{2} + 2 + \sqrt{2} + 0 \right] = \frac{\pi}{4} \left[\sqrt{2} + 1 \right]$$

Simpson:

$$I = \frac{\pi}{12} \left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi) \right]$$

$$= \frac{\pi}{12} \left[0 + 2\sqrt{2} + 2 + 2\sqrt{2} + 0 \right] = \frac{\pi}{6} \left[2\sqrt{2} + 1 \right]$$

3/ $\int_2^7 (3x-1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i).\Delta x_i$

$\Delta x = \frac{7-2}{n} = \frac{5}{n}, \quad x_i = 2 + \frac{5}{n}.i$

$\Rightarrow f(x_i) = 3.\left(2+\frac{5i}{n}\right) - 1 = \frac{15i}{n} + 5$

$I = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{15i}{n}+5\right) \frac{5}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{75i}{n^2} + \frac{25}{n}\right)$

$= \lim_{n\to\infty} \left(\left(\frac{75n(n+1)}{n^2 . 2} + \frac{25}{n}.n\right)\right) = \lim_{n\to\infty} \left(\frac{75(n+1)}{2n} + 25\right)$

$= \lim_{n\to\infty} \left[\frac{75}{2}\left(1+\frac{1}{n}\right) + 25\right] = \frac{125}{2}$

4/ $I = \int \sin^5(3z). \cos(3z). dz$

Let $u = \sin(3z) \Rightarrow du = 3\cos(3z).dz \Rightarrow \cos(3z).dz = \frac{du}{3}$

$I = \int \frac{u^5\, du}{3} = \frac{1}{3}\frac{u^6}{6} = \frac{u^6}{18} = \frac{1}{18}[\sin(3z)]^6 + C$

5/ Left endpoints: Speed × Time = Distance

$\frac{\text{Distance}}{\text{Fuel effiency}} = \text{Fuel used}$

$\text{Fuel used} = \frac{\text{Speed} \times \text{Time interval}}{\text{Fuel efficiency}}$ Time interval: $5\text{min} = \frac{1}{12}h$.

Left endpoints: $\frac{1}{12}\left[\frac{10}{15} + \frac{20}{18} + \frac{30}{21} + \frac{40}{23} + \frac{50}{24} + \frac{60}{25}\right] = 0.7857$ gallo

Right endpoints: $\frac{1}{12}\left[\frac{20}{18} + \frac{30}{21} + \frac{40}{23} + \frac{50}{24} + \frac{60}{25} + \frac{70}{26}\right] = 0.9545$ gallon

Midpoint: $\frac{1}{6}\left[\frac{20}{18}+\frac{40}{23}+\frac{60}{25}\right] = 0.8750$ gallons

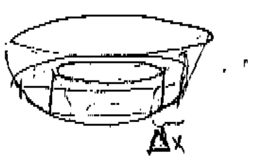
6/ $\int_0^8 t\sqrt{t+1}\,dt$. Let $u=\sqrt{t+1} \Rightarrow t = u^2-1 \Rightarrow dt = 2u\,du$.

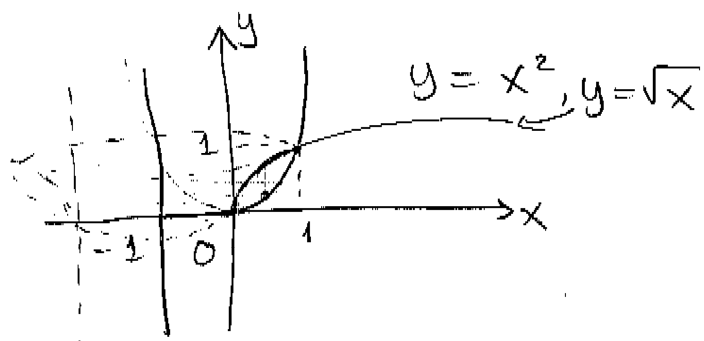
$u(0) = 1,\ u(8) = 3$.

$$I = \int_1^3 (u^2-1)u \cdot 2u\,du = 2\int_1^3 u^2(u^2-1)\,du = 2\int_1^3 (u^4-u^2)\,du$$

$$= 2\left[\frac{u^5}{5}-\frac{u^3}{3}\right]_1^3 = 2\left[\frac{3^5}{5}-\frac{3^3}{3}-\frac{1}{5}+\frac{1}{3}\right] = \frac{1192}{15}$$

7/ Cylindrical shells.





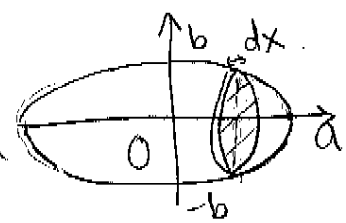
Circumference : $2\pi(x+1)$
Thickness : Δx
Height : $\sqrt{x}-x^2$.

$$\Rightarrow V = \int_0^1 2\pi(x+1)(\sqrt{x}-x^2)\,dx$$

$$= 2\pi\int_0^1 (x^{3/2}+x^{1/2}-x^3-x^2)\,dx = 2\pi\left[\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}-\frac{x^4}{4}-\frac{x^3}{3}\right]$$

$$= 2\pi\left[\frac{2}{5}+\frac{2}{3}-\frac{1}{4}-\frac{1}{3}\right] = \frac{29\pi}{30}$$

8/ $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$



$y=0 \Rightarrow x = \pm a$; $x=0 \Rightarrow y = \pm b$.

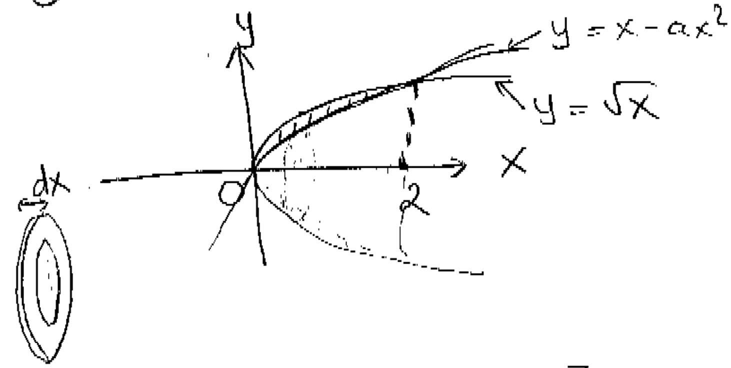
$$A = \pi \cdot y^2\Delta x = \pi\ b^2\left(1-\frac{x^2}{a^2}\right)\Delta x$$

$$\Rightarrow V = \int^a \pi b^2\left(1-\frac{x^2}{a^2}\right)dx = \pi b^2\int^a \left(1-\frac{1}{a^2}\cdot x^2\right)dx$$

$$V = \pi b^2 \cdot \left[x - \frac{1}{3a^2} \cdot x^3\right]_0^a = \pi b^2\left[a - \frac{a^3}{3a^2}\right] = \pi b^2 \cdot \left[a - \frac{a}{3}\right] \quad 4$$

$$= \pi b^2 \frac{2a}{3} \Rightarrow V = \frac{2\pi a b^2}{3}$$

9/



Intersect $\sqrt{x} = x - ax^2$

$x = 0$ and $x = 2$

Area $= \pi.(\sqrt{x})^2 - \pi.(x - ax^2)^2$

$$= \pi\left[x - (x^2 - 2ax^3 + a^2x^4)\right] = \pi\left[-a^2x^4 + 2ax^3 - x^2 + x\right]$$

Volume $V = \int_0^2 \pi(-a^2x^4 + 2ax^3 - x^2 + x)\,dx$

$$= \pi\left[-a^2 \cdot \frac{x^5}{5} + \frac{ax^4}{2} - \frac{x^3}{3} + \frac{x^2}{2}\right]_0^2$$

$$= \pi\left[-\frac{32a^2}{5} + 8a - \frac{8}{3} + 2\right] = \pi\left[-\frac{32a^2}{5} + 8a - \frac{2}{3}\right]$$

$$= 2\pi\left[-\frac{16}{5} \cdot \frac{1}{16}(2-\sqrt{2})^2 + 4 \cdot \frac{1}{4}(2-\sqrt{2}) - \frac{1}{3}\right]$$

$$= 2\pi\left[-\frac{(2-\sqrt{2})^2}{5} + (2-\sqrt{2}) - \frac{1}{3}\right]$$

$$= 2\pi\left[\frac{5(2-\sqrt{2}) - (4 + 2 - 4\sqrt{2})}{5} - \frac{1}{3}\right]$$

$$= 2\pi\left[\frac{10 - 5\sqrt{2} - 6 + 4\sqrt{2}}{5} - \frac{1}{3}\right] = 2\pi\left[\frac{4 - \sqrt{2}}{5} - \frac{1}{3}\right]$$

$$= \frac{2\pi(12 - 3\sqrt{2} - 5)}{15} = \frac{2\pi(7 - 3\sqrt{2})}{15} = \frac{14\pi}{15} - \frac{2\sqrt{2}\pi}{5}$$

(10): $\int_{-3}^{2} e^{-4x}\,dx$ (5)

$E_T \leqslant \frac{K(b-a)^3}{12n^2}$ $f''(x) \leqslant K$

$f'(x) = -4 \cdot e^{-4x} \Rightarrow |f''(x) = 16e^{-4x}| \leqslant |f''(-3)| = |16e^{12}|$

Choose $K = 16 \cdot e^{12}$

i. $\frac{16 \cdot e^{12} 125}{12n^2} \leqslant 0.00001 \Rightarrow n \doteq 1,646,992$

$E_N \leqslant \frac{K(b-a)^3}{24n^2} \Leftrightarrow \frac{16 \cdot e^{12} 125}{24n^2} \leqslant 0.00001 \Rightarrow n = 1,164,599.$

$E_S \leqslant \frac{K(b-a)^5}{180n^4}$ $|f^{(4)}(x)| \leqslant K$. $f^{(3)}(x) = -64e^{-4x}$

$\Rightarrow f^{(4)}(x) = 256e^{-4x}$

$= |f^{(4)}(x)| \leqslant f^{(4)}(-3) = 256e^{12} = K$

$\frac{5^5 \, e^{12} \, 125}{180n^4} \leqslant 0.00001 \Rightarrow n \geqslant 2437.$

$\Rightarrow n = 2438.$

(12). $\int_{1}^{\infty} \frac{\sin x + 3}{\sqrt{x}}\,dx \leqslant \underbrace{\int^{\infty} \frac{3}{\sqrt{x}}\,dx}$: diverge by p-test

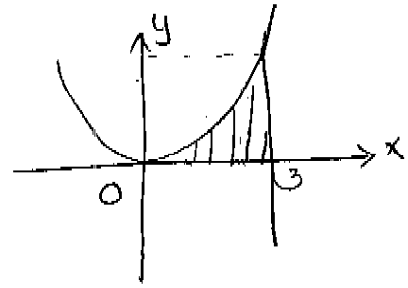
divergent

(13). $y = \frac{x^2}{2} - \frac{\ln x}{4}$ on $2 \leqslant x \leqslant 4 \Rightarrow y' = x - \frac{1}{4x} \Rightarrow (y')^2 = x^2 + \frac{1}{16x^2} - \frac{1}{2}$

$L = \int_{2}^{4} \sqrt{1 + x^2 + \frac{1}{16x^2} - \frac{1}{2}}\,dx = \int_{2}^{4} \left|x + \frac{1}{4x}\right| dx$

$= \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left[8 + \frac{\ln 4}{4} - \left(2 + \frac{\ln 2}{4}\right)\right] = 6 + \frac{1}{4}\ln 2$

14/ Centroid: $y = x^2$, $y = 0$ and $x = 3$:



$$A = \int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = 9.$$

$$\bar{x} = \frac{1}{A}\int_0^3 x \cdot x^2\,dx = \frac{1}{9}\left[\frac{x^4}{4}\right]_0^3 = \frac{1}{36}\cdot 81$$

$$\bar{x} = \frac{81}{36} = \frac{9}{4}$$

$$\bar{y} = \frac{1}{A}\int_0^3 \frac{1}{2}x^4\,dx = \frac{1}{18}\int_0^3 x^4\,dx = \frac{1}{90}\left[x^5\right]_0^3 = \frac{243}{90} = \frac{27}{10}.$$

Centroid $\left(\frac{9}{4}, \frac{27}{10}\right)$

15/ $y' = y^4 - 6y^3 + 5y^2 = y^2(y^2 - 6y + 5) = y^2(y-1)(y-5)$

+		+		−		+
	0		1		5	

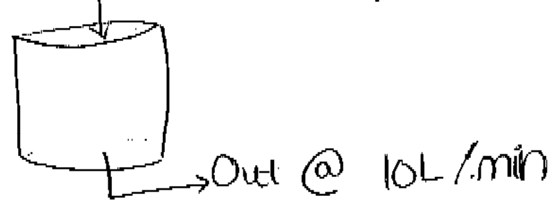
a/ Constant solutions of equation

$y = 0$, $y = 1$, $y = 5$

b/ y is increasing when: $y \leqslant 1$ & $y \geqslant 5$.

y is decreasing when $1 < y < 5$

16/ 1,000 L brine - 15 kg salt



$A(t)$ = amount of salt in tank at time t.

$$\frac{dA}{dt} = \text{rate in} - \text{rate out} = 0.025 \cdot 10\,\frac{kg}{min} - \frac{A}{1{,}000}\cdot 10\,\frac{kg}{min} = 0.25 - \frac{A}{100}$$

$$\frac{dA}{dt} = \frac{25 - A}{100} \Leftrightarrow \int \frac{dA}{25-A} = \int \frac{dt}{100} \Leftrightarrow -\ln|25 - A| = \frac{1}{100}\cdot t + c_1$$

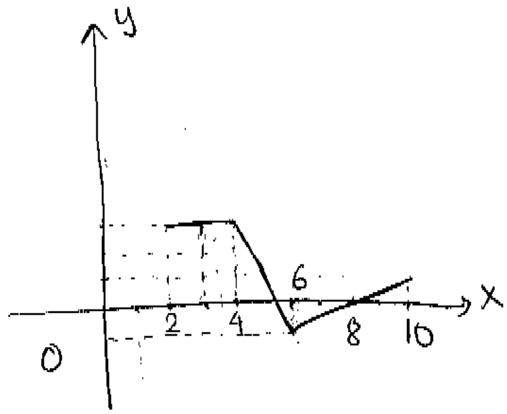
$$\Leftrightarrow \ln|25 - A| = -0.01t + c_2 \Leftrightarrow 25 - A = c_3 \cdot e^{-0.01t} \Rightarrow A = 25 - c_3 \cdot e^{-0.01t}$$

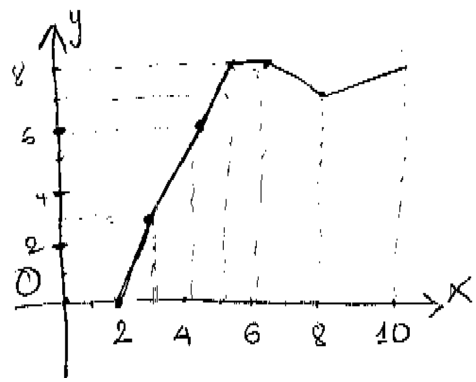
$$A = 25 + c_4 \cdot e^{-0.01t}$$

$A(0) = 25 + c_4 = 15 \Rightarrow c_4 = -10$.

$A(60) = 25 - 10e^{-0.01(60)} = 19.512\text{kg}$

11/





$x = 2 \Rightarrow g(x) = 0$, $x = 3 \Rightarrow g(x) = 3$
$x = 4 \Rightarrow g(x) = 6$, $x = 5 \Rightarrow g(x) = 8$
$x = 6 \Rightarrow g(x) = 8$; $x = 8 \Rightarrow g(x) = 7$
$x = 10 \Rightarrow g(x) = 8$.

17/ $\frac{dP}{dh} = kP \Leftrightarrow \int \frac{dP}{P} = \int k\,dh \Leftrightarrow \ln|P| = k.h + c_1$

$\Rightarrow P = c_2 . e^{kh} = 101.3e^{-0.00015h}$.

$P(0) = c_2 = 101.3$; $P(1{,}000) = 101.3\, e^{1000k} = 87.14$
$\Rightarrow 1000k = \ln\frac{87.14}{101.3} \Rightarrow k = -0.0001$

a/ $P(3{,}000) = 101.3\, e^{(-0.00015)(3000)} = 64.59$

b/ $P(6{,}187) = 101.3\, e^{(-0.00015)(6187)} = 40.05$

$\boxed{\text{Area of a circle}}$

$$A = 4\int_0^r \sqrt{r^2 - x^2}\, dx \qquad \begin{aligned} x &= r\sin\theta \\ dx &= r\cos\theta\, d\theta \end{aligned}$$

$$= 4\int_0^{\pi/2} r\cos\theta \cdot r\cos\theta\, d\theta$$

$$= 4r^2 \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= 4r^2 \left[\frac{1}{2}\theta + \frac{\overset{\sin}{\cancel{\cos}}\, 2\theta}{4}\right]_0^{\pi/2}$$

$$= 4r^2 \left(\frac{\pi}{4} + \frac{1}{4}0 - 0 - 0\right)$$

$$= \pi r^2$$

$\boxed{\text{circumference of a circle}}$

$$y = \sqrt{r^2 - x^2}$$

$$y' = \frac{-x}{\sqrt{r^2-x^2}}$$

$$C = 4 \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx$$

$$= 4 \int_0^r \frac{r}{\sqrt{r^2-x^2}} \, dx$$

$$x = r\sin\theta$$

$$dx = r\cos\theta \, d\theta$$

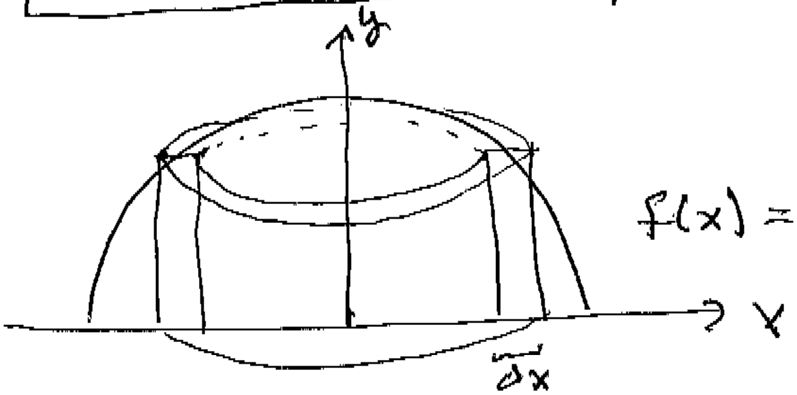
$$= 4 \int_0^{\pi/2} \frac{r}{r\cos\theta} \cdot r\cos\theta \, d\theta$$

$$= 4 \Big[r\theta \Big]_0^{\pi/2}$$

$$= 4r\frac{\pi}{2}$$

$$= 2\pi r.$$

$\boxed{\text{Volume of a sphere.}}$



$f(x) = \sqrt{r^2 - x^2}$

$dV = 2\pi x f(x)\,dx$

$V = 2\int_0^r 2\pi x \sqrt{r^2 - x^2}\,dx$

$u = r^2 - x^2$

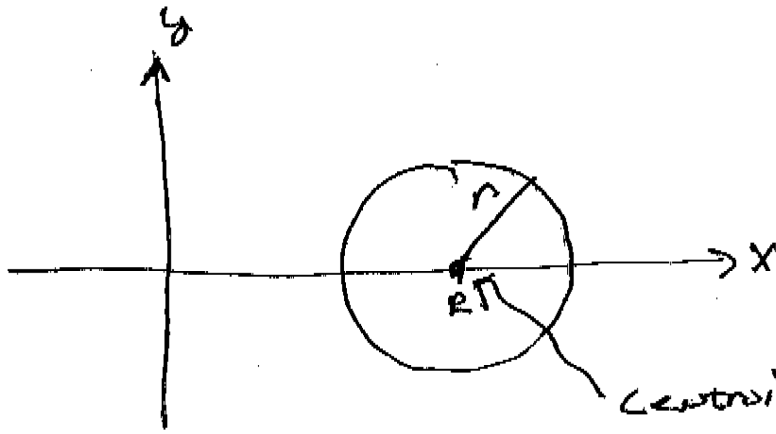
$du = -2x\,dx$

$= -2\pi \int_{r^2}^{0} u^{1/2}\,du$

$= 2\pi \frac{2}{3} u^{3/2} \Big|_0^{r^2}$

$= \frac{4}{3}\pi r^3.$

Volume of a torus

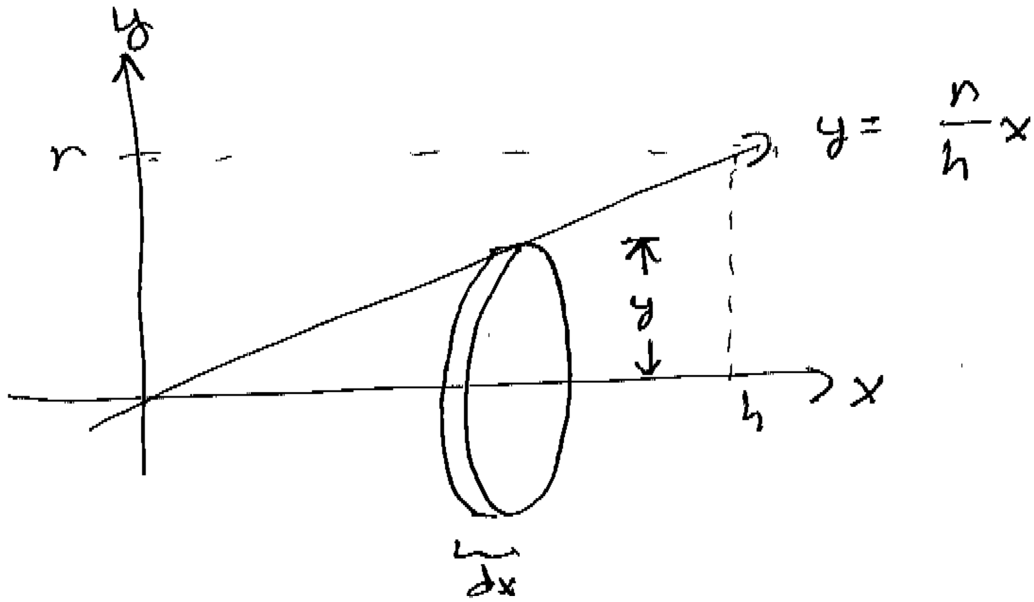


Area of the region is πr^2.

$$V = \pi r^2 \cdot 2\pi R$$

$$= 2\pi^2 r^2 R$$ by the Thm of Pappus.

volume of a cylindrical cone.



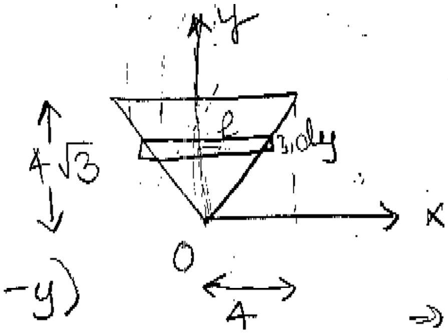
$$V = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx$$

$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi r^2 h}{3}$$

19/ $\rho = 840 \, kg/m^3$

$$\text{Area} = \frac{2 \cdot y}{\sqrt{3}} \cdot dy$$



$$\frac{l}{4} = \frac{y}{4\sqrt{3}}$$

$$\text{Pressure} = \rho \cdot g \cdot d = \rho \cdot g(4\sqrt{3} - y)$$

$$\Rightarrow l = \frac{4y}{4\sqrt{3}} = \frac{y}{\sqrt{3}}$$

$$F = \int_0^{4\sqrt{3}} \rho \cdot g (4\sqrt{3} - y) \frac{2y}{\sqrt{3}} dy = \frac{(840)(9.8) \cdot 2}{\sqrt{3}} \int_0^{4\sqrt{3}} (4\sqrt{3} - y) y \, dy$$

$$= \frac{16464}{\sqrt{3}} \int_0^{4\sqrt{3}} (4\sqrt{3}y - y^2) dy = \frac{16464}{\sqrt{3}} \left[2\sqrt{3}y^2 - \frac{y^3}{3} \right]_0^{4\sqrt{3}}$$

$$= \frac{16464}{\sqrt{3}} \left[2\sqrt{3} \cdot 16 \cdot 3 - \frac{64 \cdot 3\sqrt{3}}{3} \right] = \frac{16464}{\sqrt{3}} \left[96\sqrt{3} - 64\sqrt{3} \right] = 526848 \, PA$$