7.3 #32b

Let $x = a\sinh t$

$dx = a\cosh t\, dt$

$\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

$\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$

$$I = \int \frac{\cancel{a^2}\sinh^2 t \cdot \cancel{a}\cosh t\, dt}{\cancel{a^3}\sqrt{\sinh^2 t + 1}^{\,3}}$$

$$= \int \frac{\sinh^2 t \cdot \cosh t}{\cosh^3 t}\, dt$$

$$= \int \frac{\sinh^2 t}{\cosh^2 t}\, dt$$

$$= \int \tanh^2 t\, dt$$

$$= \int (1 - \operatorname{sech}^2 t)\, dt$$

$$= t - \int \operatorname{sech}^2 t\, dt$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) - \tanh\left(\sinh^{-1}\left(\frac{x}{a}\right)\right) + C$$

We need to find $\sinh^{-1}(x)$

$$y = \sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$$\Rightarrow \quad 2y = e^{x} - \frac{1}{e^{x}} = \frac{(e^{x})^{2} - 1}{e^{x}}$$

$$\Rightarrow \quad 0 = (e^{x})^{2} - 2y(e^{x}) - 1$$

$$\Rightarrow \quad e^{x} = \frac{\cancel{2}y \pm \sqrt{\cancel{4}y^{2} - \cancel{4}(1)(-1)}}{\cancel{2}}$$

$$= y \pm \sqrt{y^{2}+1}$$

$$x = \ln\left(y \oplus \sqrt{y^{2}+1}\right) \qquad -\ln \bigcirc = \ln \bigcirc^{-1}$$

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^{2}+1}\right)$$

$$I = \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^{2}+1}\right) - \frac{e^{\ln(\ )} - e^{-\ln(\ )}}{e^{\ln(\ )} + e^{-\ln(\ )}} + C$$

$$I = \ln\left(\frac{x}{a} + \frac{1}{a}\sqrt{x^2+a^2}\right) - \frac{\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right)} \cdot \frac{x-\sqrt{x^2+a^2}}{x-\sqrt{x^2+a^2}}}{\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right) + \frac{1}{\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right)} \cdot \frac{x-\sqrt{x^2+a^2}}{x-\sqrt{x^2+a^2}}}$$

$$- \frac{\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right) + \frac{a\left(x-\sqrt{x^2+a^2}\right)}{a^2}}{\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right) - \frac{a\left(x-\sqrt{x^2+a^2}\right)}{a^2}}$$

$$- \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}$$

$$I = \underbrace{\ln\left(\frac{x}{a} + \frac{1}{a}\sqrt{x^2+a^2}\right)} - \frac{x}{\sqrt{x^2+a^2}} + C$$

$$\ln\left(\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right)\right)$$

$$\underbrace{\ln\frac{1}{a}}_{\text{incl. in "}C\text{"}} + \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$= \ln\left(x+\sqrt{x^2+a^2}\right) - \frac{x}{\sqrt{x^2+a^2}} + C.$$

$$\tanh(\sinh^{-1}(x)) = \frac{x + \cancel{\sqrt{x^2+1}} + (x - \cancel{\sqrt{x^2+1}})}{\cancel{x} + \sqrt{x^2+1} - (\cancel{x} - \sqrt{x^2+1})}$$

$$= \frac{2x}{2\sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2+1}}$$

so, $\tanh\left(\sinh^{-1}\left(\frac{x}{a}\right)\right) = \dfrac{x/a}{\sqrt{\left(\frac{x}{a}\right)^2+1}} = \dfrac{x}{\sqrt{x^2+a^2}}$