100	90's	80's	70's	60's	< 60%
7	6	3	2	2	6

Test 3a
Dusty Wilson
Math 220

high 100%
$\bar{x}$ 79.9%
Med 87.5%

Name: key

A man has one pair of rabbits at a certain place entirely surrounded by a wall. We wish to know how many pairs will be bred from it in one year, if the nature of these rabbits is such that they breed every month one other pair and begin to breed in the second month after their birth.

Leonardo Pisano Fibonacci
1170-1250 (Italian traveler and mathematician)

No work = no credit
No calculators

1.) (4 pts) Find the eigenvalue(s) of $A=\begin{bmatrix}5 & -5\\ 2 & -1\end{bmatrix}$

solve $0=\begin{vmatrix}5-\lambda & -5\\ 2 & -1-\lambda\end{vmatrix}$

$=(5-\lambda)(-1-\lambda)+10$

$=\lambda^2-4\lambda+5$

$\Rightarrow \lambda = \dfrac{4\pm\sqrt{16-4(1)(5)}}{2(1)}$

$=\dfrac{4\pm 2i}{2}$

$=2\pm i$

2.) (4 pts) Find the eigenvalue(s) of $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 3 & 1\\ 0 & 0 & 4\end{bmatrix}$ given that $\lambda=4$ is an eigenvalue.

solve $0=\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 3-\lambda & 1\\ 0 & 0 & 4-\lambda\end{vmatrix}$

$=(4-\lambda)\begin{vmatrix}3-\lambda & -1\\ -1 & 3-\lambda\end{vmatrix}$

$=(4-\lambda)\left[(3-\lambda)^2-1\right]$

$=(4-\lambda)\left[\lambda^2-6\lambda+9-1\right]$

$=(4-\lambda)(\lambda-4)(\lambda-2)$

$(4-\lambda)(\lambda^2-6\lambda+8)$
$=4\lambda^2-24\lambda+32$
$-\lambda^3+6\lambda^2-8\lambda$
$=-\lambda^3+10\lambda^2-32\lambda+32$

$\Rightarrow \lambda=2$ (alg. mult 1) and $\lambda=4$ (alg. mult 2).

3.) (7 pts) Consider $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

a.) (1 pt) Verify that $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector and find the corresponding eigenvalue.

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{so } \lambda = 1.$$

b.) (4 pts) Find the eigenvector(s) associated with the eigenvalue $\lambda = 3$.

I need to rref $A - 3I$.

$$A - 3I = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 - R_1 \to R_2$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -1R_1 \to R_1$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so } E_{\lambda=3} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

c.) (1 pt) $\dim(E_{\lambda=3}) =$ <u>2</u>

d.) (1 pt) Is there an eigenbasis for A? Yes No (Circle one).

Test 3b
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Warm-ups (1 pt each)[1]: $1+1=$ 2 $\quad -1^2=$ -1 $\quad \vec{e}_1 \cdot \vec{e}_2=$ 0

1.) (1 pt) List of the next five terms of the Fibonacci sequence beginning 1, 1, 2

1, 1, 2, 3, 5, 8, 13, 21, ...

2.) (4 pts) Decide if the matrix $A=\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable. If possible, find an invertible S and a diagonal D such that $A=SDS^{-1}$.

$\lambda = 2, 3$

$E_{\lambda=3} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$E_{\lambda=2} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

so $A = SDS^{-1}$ w/ $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

[1] In the warm-ups, $\vec{e}_i$ refers to the standard basis vector in $\mathbb{R}^2$.

3.) (4 pts) If $\vec{x}_0 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $A = SDS^{-1}$ where $S = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$, find a closed form expression for $\vec{x}(t) = A^t \vec{x}_0$.

$$S^{-1}\vec{x}_0 = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

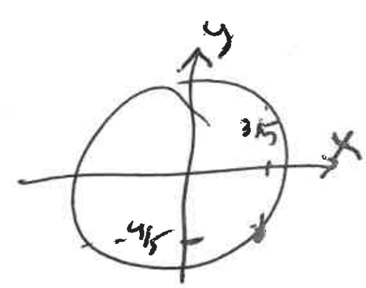
$$\Rightarrow D^t S^{-1} x_0 = \begin{bmatrix} 5^t & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5^t \\ 0 \end{bmatrix}$$

$$\Rightarrow S D^t S^{-1} \vec{x}_0 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5^t \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 5^t \\ 2 \cdot 5^t \end{bmatrix}$$

and $\vec{x}(t) = A^t \vec{x}_0 = 5^t \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

4.) (4 pts) Consider the rotation-scaling matrix $B = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$. Find the angle of rotation (degrees or radians to two decimal places) and the scaling factor.

$$r = \sqrt{3^2 + 4^2} = 5$$



$r\cos\theta = 3$

AND $r\sin\theta = -4$

4th quadrant

so $\tan\theta = -\frac{4}{3}$

$$\Rightarrow \theta = -\tan^{-1}\left(\frac{4}{3}\right)$$

$= -0.93$ rad. or $-53.13°$

5.) (4 pts) Show $A=\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}$ is similar to the rotation-scaling matrix $B=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$.

The eigenvals would have to be $2 \pm i$

Find the eigenvec w/ $\lambda = 2+i$

$$A-(-1+2i)I = \begin{bmatrix}3-(2+i) & 1\\ -2 & 1-(3+i)\end{bmatrix}$$

$$= \begin{bmatrix}1-i & 1\\ -2 & -1-i\end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix}-2 & -1-i\\ 1-i & 1\end{bmatrix} \quad -\tfrac{1}{2}R_1 \to R_1$$

$$= \begin{bmatrix}1 & \frac{1}{2}+\frac{i}{2}\\ 1-i & 1\end{bmatrix} \quad R_2-(1-i)R_1 \to R_1$$

Scratch

$$1-(1-i)\left(\tfrac{1}{2}+\tfrac{i}{2}\right) = 1-\left(\tfrac{1}{2}-\tfrac{i^2}{2}\right) = 0$$

$$= \begin{bmatrix}1 & \frac{1}{2}+\frac{i}{2}\\ 0 & 0\end{bmatrix}$$

So the eigenvec is $\begin{bmatrix}-\frac{1}{2}-\frac{i}{2}\\ 1\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix} \pm i\begin{bmatrix}-\frac{1}{2}\\ 0\end{bmatrix}$

So $S = \begin{bmatrix}-\frac{1}{2} & -\frac{1}{2}\\ 0 & 1\end{bmatrix}$ and $AS = SB$

verify.

6.) (6 pts) (True or False) Answer the following. 1 point per problem is for an explanation/justification:

a.) (True or False) If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is also an eigenvector of $A+B$.

suppose $A\vec{v} = \lambda_A \vec{v}$ and $B\vec{v} = \lambda_B \vec{v}$

$(A+B)\vec{v} = A\vec{v} + B\vec{v} = \lambda_A \vec{v} + \lambda_B \vec{v} = (\lambda_A + \lambda_B)\vec{v}$

$\therefore$ $\vec{v}$ is an eigenvec of $A+B$.

b.) (True or False) All diagonalizable matrices are invertible.

example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = I \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} I^{-1}$

c.) (True or False) If two n x n matrices A and B are diagonalizable, then $A+B$ must be diagonalizable as well.

A & B diagonalizable

example: $\underset{A+B}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}} + \underset{B}{\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}}$

$A+B$ has only 1 L.I. eigenvec. so not diagonalizable

7.) (4 pts) Prove that if A is similar to B, then matrices A and B have the same characteristic polynomial.

proof.

Let similar A and B be given.

$\Rightarrow \exists\, S$ s.t. $A = SBS^{-1}$

$$\det(A - \lambda I) = \det(SBS^{-1} - \lambda SS^{-1})$$
$$= \det(S(B - \lambda I)S^{-1})$$
$$= \det(S)\det(B - \lambda I)\det(S^{-1})$$
$$= \det(B - \lambda I).$$

$\therefore$ They have the same characteristic polynomial.

Q.E.D.