100+	90's	80's	70's	60's	< 60
3	4	6	6	2	7

Test 2a
Dusty Wilson
Math 220

high 107.8%
$\bar{x} = 70.6\%$
med = 71.8%

Name: Key

Unfortunately what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.

Evariste Galois
1811 – 1832 (French mathematician)

No work = no credit
No calculators

1.) (4 pts) Find the determinant of $A=\begin{bmatrix}-1 & -1 & 2 & 3\\ 2 & -3 & -3 & 1\\ 3 & 2 & 1 & 3\\ 1 & 2 & 0 & -1\end{bmatrix}$. Hint: The result is between ± 100.

$$\det(A) = -1\begin{vmatrix}-1 & 2 & 3\\ -3 & -3 & 1\\ 2 & 1 & 3\end{vmatrix} + 2\begin{vmatrix}-1 & 2 & 3\\ 2 & -3 & 1\\ 3 & 1 & 3\end{vmatrix} - 1\begin{vmatrix}-1 & -1 & 2\\ 2 & -3 & -3\\ 3 & 2 & 1\end{vmatrix}$$

#1 #2 #3

$$= -1(41) + 2(37) - 1(34)$$

$$= -41 + 74 - 34$$

$$= -1$$

#1 $\begin{vmatrix}-1 & 2 & 3\\ -3 & -3 & 1\\ 2 & 1 & 3\end{vmatrix} = -1\begin{vmatrix}-3 & 1\\ 1 & 3\end{vmatrix} - 2\begin{vmatrix}-3 & 1\\ 2 & 3\end{vmatrix} + 3\begin{vmatrix}-3 & -3\\ 2 & 1\end{vmatrix}$

$$= -1(-10) - 2(-11) + 3(3) = 41$$

#2 $\begin{vmatrix}-1 & 2 & 3\\ 2 & -3 & 1\\ 3 & 1 & 3\end{vmatrix} = -1\begin{vmatrix}-3 & 1\\ 1 & 3\end{vmatrix} - 2\begin{vmatrix}2 & 1\\ 3 & 3\end{vmatrix} + 3\begin{vmatrix}2 & -3\\ 3 & 1\end{vmatrix}$

$$= -1(-10) - 2(3) + 3(11) = 37$$

#3 $\begin{vmatrix}-1 & -1 & 2\\ 2 & -3 & -3\\ 3 & 2 & 1\end{vmatrix} = -1\begin{vmatrix}-3 & -3\\ 2 & 1\end{vmatrix} + 1\begin{vmatrix}2 & -3\\ 3 & 1\end{vmatrix} + 2\begin{vmatrix}2 & -3\\ 3 & 2\end{vmatrix}$

$$= -1(3) + 1(11) + 2(13) = 34$$

2.) (4 pts) Prove the given theorem.

Theorem: Consider $\vec{v}_1,\ldots,\vec{v}_m$ in a subspace V of $\mathbb{R}^n$. Then $\vec{v}_1,\ldots,\vec{v}_m$ are a basis for V iff all vectors $\vec{v} \in V$ can be expressed as a unique linear combination of $\vec{v}_1,\ldots,\vec{v}_m$.

proof

($\Rightarrow$) Assume $\vec{v}_1, \ldots, \vec{v}_m$ are a basis for V.

suppose $\vec{v} \in V$ cannot be expressed as a unique linear combination of $\vec{v}_1, \ldots, \vec{v}_m$.

$\Rightarrow \exists\ c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ s.t.

$c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \vec{v}$ and $d_1\vec{v}_1, \ldots, d_m\vec{v}_m = \vec{v}$

$\Rightarrow (c_1 - d_1)\vec{v}_1 + \ldots + (c_m - d_m)\vec{v}_m = \vec{0}$

since $\vec{v}_1, \ldots, \vec{v}_m$ is a basis, this equation has only the trivial soln so $c_1 = d_1, \ldots, c_m = d_m \Rightarrow\Leftarrow$

$\Rightarrow$ all $\vec{v} \in V$ can be expressed as a unique linear combination of $\vec{v}_1, \ldots, \vec{v}_m$.

($\Leftarrow$) Assume all vectors $\vec{v} \in V$ can be expressed as a unique lin. comb of $\vec{v}_1, \ldots, \vec{v}_m$.

clearly $\vec{v}_1, \ldots, \vec{v}_m$ span V.

$c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \vec{0}$ has only the trivial solution.

$\Rightarrow \vec{v}_1, \ldots, \vec{v}_m$ are L.I. and form a basis.

Q.E.D.

Test 2b
Dusty Wilson
Math 220

No work = no credit

Name: Key

Unfortunately what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.

Evariste Galois
1811 – 1832 (French mathematician)

Warm-ups (1 pt each): $\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix} = \begin{bmatrix}3 & 4\\6 & 8\end{bmatrix}$ $\quad \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}11\end{bmatrix}$ $\quad AA^{-1} = I$

1.) (1 pt) According to Galois, how do authors most hurt their readers? Answer using complete English sentences.

Authors make everything look too easy.

2.) (3 pts) Define the following:

a.) What does it mean if $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ are linearly independent?

No $\vec{v}_i$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{i-1}$

b.) What is the span of $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$.

This is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$

c.) How do we know if $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ are a basis for a subspace V of $\mathbb{R}^n$?

$\vec{v}_1, \ldots, \vec{v}_m$ are a basis for V if they span V and are L.I.

3.) (4 pts) According to our text, a subset W of the vector space $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it includes a zero and is closed under addition and scalar multiplication.

Is the set $W = \left\{ \begin{bmatrix}x\\y\end{bmatrix} : x^2 = x + y \right\}$ a subspace. Justify your answer.

NO $\quad \begin{bmatrix}1\\0\end{bmatrix} \in W$ but $2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \notin W$

so W isn't closed under scalar mult.

4.) (6 pts) Answer the following and justify your answer.

a.) True or False, there exists a 5x4 matrix whose image consists of all of $\mathbb{R}^5$

rank ≤ 4

b.) True or False, If A is a 5 x 6 matrix of rank 4, then the nullity of A is 1.

rank + nullity = 6

c.) True or False, if A and B are both n x n matrices, and vector $\vec{v}$ is in the kernel of ~~both A and~~ B, then $\vec{v}$ must be in the kernel of matrix AB as well.

$$(AB)\vec{v} = A(B\vec{v}) = A\vec{0} = \vec{0}$$

5.) (10 pts) Consider the linear transformation $T(\vec{x}) = A\vec{x}$ such that $T(\vec{v}_1) = \vec{v}_1 + 3\vec{v}_2$ and $T(\vec{v}_2) = 2\vec{v}_1 + 7\vec{v}_2$ where $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

a.) Find the matrix B of the linear transformation

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

b.) If $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $[\vec{x}]_B$

$$S = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \Rightarrow S^{-1}\vec{x} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}_B}_{[\vec{x}]_B}$$

c.) For the given $\vec{x}$, find $[T(\vec{x})]_B$

$$B[\vec{x}]_B = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ -4 \end{bmatrix}_B}_{[T(\vec{x})]_B}$$

d.) For the given $\vec{x}$, find $T(\vec{x})$. Hint: One component is -17.

$$S[T(\vec{x})]_B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ -4 \end{bmatrix} = \underbrace{\begin{bmatrix} -6 \\ -17 \end{bmatrix}}_{T(\vec{x}).}$$

6.) (5 pts) Consider the matrix $A = \begin{bmatrix} 6 & 3 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & 1 & 2 & -2 & -3 \\ 3 & 6 & 1 & 1 & 4 \end{bmatrix}$

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & 1/4 & 1 \\ 0 & 0 & 1 & -5/4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a.) Find a basis for the image of A.

basis: $\left\{ \begin{bmatrix} 6 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

b.) Find the kernel of A.

$\ker(A) = \text{span}\left(\begin{bmatrix} -1/4 \\ -1/4 \\ 5/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$

c.) rank(A) = 3 and nullity(A) = 2

7.) (5 pts) For the matrix $A_{n x n}$, there are at least 9 statements equivalent to, "A is invertible." List at least five of them. List more for extra credit.

i.) A is invertible.	vi.) $\ker(A) = \vec{0}$
ii.) $\text{rank}(A) = n$	vii.) (0.25 pt extra credit) cols are L.I.
iii.) $\text{nullity}(A) = 0$	viii.) (0.5 pt extra credit) $\text{rref}(A) = I$
iv.) $\det(A) \neq 0$	ix.) (0.75 pt extra credit) cols span $\mathbb{R}^n$.
v.) $\text{im}(A) = \mathbb{R}^n$	x.) (1 pt extra credit) $A\vec{x} = \vec{b}$ has a unique soln $\forall \vec{b} \in \mathbb{R}^n$.

0 is not an eigenvalue of A.

cols of A are a basis for $\mathbb{R}^n$.