Test 1 – Part A
Dusty Wilson
Math 220

Name: key

An elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline
1908-1992 (American mathematician)

No work = no credit
No Graphing Calculators

Warm-ups (1 pt each): $\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix}=\begin{bmatrix}3 & 6\\4 & 8\end{bmatrix}$ $\quad I\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$ $\quad \vec{e}_2\cdot\vec{e}_2=$ 1

1.) (1 pts) According to ~~Kline~~ Gauss (above), how should a good proof be written? Answer using complete English sentences.

Gauss was a perfectionist who liked to be concise.

2.) (8 pts) Consider $A=\begin{bmatrix}3 & 5\\2 & 4\end{bmatrix}$. Find A^{-1} (if it exists) **<u>using row reduction</u>**. If it doesn't exist, write the letters of my first name in alphabetical order.

$$\left[\begin{array}{cc|cc}3 & 5 & 1 & 0\\2 & 4 & 0 & 1\end{array}\right] R_1-R_2\to R_1$$

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & -1\\2 & 4 & 0 & 1\end{array}\right] R_2-2R_1\to R_2$$

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & -1\\0 & 2 & -2 & 3\end{array}\right] \tfrac{1}{2}R_2\to R_2$$

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & -1\\0 & 1 & -1 & \frac{3}{2}\end{array}\right] R_1-R_2\to R_1$$

$$\left[\begin{array}{cc|cc}1 & 0 & 2 & -5/2\\0 & 1 & -1 & 3/2\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}2 & -5/2\\-1 & 3/2\end{bmatrix}$$

3.) (4 pts) If $A=\begin{bmatrix}2 & 1\\-3 & 0\\0 & 5\end{bmatrix}$ and $B=\begin{bmatrix}3 & 2\\1 & -2\\-6 & 0\end{bmatrix}$, find $2A-B$

$$2A-B=\begin{bmatrix}1 & 0\\-7 & 2\\6 & 10\end{bmatrix}$$

-2 if no matrices.

4.) (8 pts) Solve the system of linear equations using Gauss-Jordan Elimination

$$\begin{cases} 2x_1 + x_2 + 6x_3 = 11 \\ \quad\;\; 2x_2 + 4x_3 = 6 \\ x_1 \qquad\;\; + 3x_3 = 9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 11 \\ 0 & 2 & 4 & 6 \\ 1 & 0 & 3 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 9 \\ 0 & 2 & 4 & 6 \\ 2 & 1 & 6 & 11 \end{array}\right] \; R_3 - 2R_1 \to R_3$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 9 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 0 & -7 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 9 \\ 0 & 1 & 0 & -7 \\ 0 & 2 & 4 & 6 \end{array}\right] \; R_3 - 2R_2 \to R_3$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 9 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 4 & 20 \end{array}\right] \xrightarrow{\frac{1}{4}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 9 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 5 \end{array}\right] \; R_1 - 3R_3 \to R_1$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad \text{so} \quad \vec{x} = \begin{bmatrix} -6 \\ -7 \\ 5 \end{bmatrix}$$

5.) (4 pts) Evaluate $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 5 & 2 & -4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 6 & 0 & 1 \\ 2 & -1 & 0 \\ 4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 2 & 10 \\ 18 & 2 & 1 \end{bmatrix}$

$\frac{1}{4}$ if a dim, error.

60's	70's	80's	90's
3	7	4	4

mean 79.4%
med. 78.6%
high 94.5%

12:15
12:35

Test 1 – Part B
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Math 220

Name: KEY

No work = no credit

1.) (9pts) Suppose $A\vec{x} = \vec{b}$ and $\operatorname{rref}\left(\left[A \mid \vec{b}\right]\right)$ is as given below. Give $\vec{x}$ in vector form.

a.) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$

no solution
or
inconsistent

b.) $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$

$$\vec{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

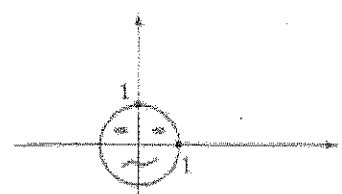
c.) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

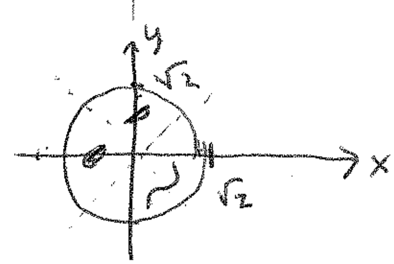
$$\vec{x} = \begin{bmatrix} 2 - 3t \\ 6 + 5t \\ t \end{bmatrix} \leftarrow \text{both ok}$$

$$= \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

2.) (8 pts) Consider the circular face. Draw a sketch showing the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ on this face. Make sure to clearly indicate the scale.

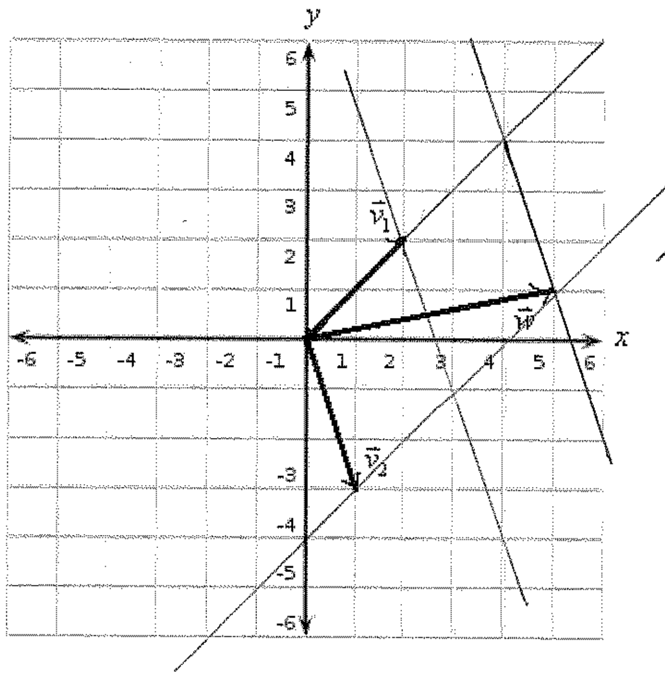
$$A = \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\text{expand by } \sqrt{2}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\text{rotate ccw by } 45^\circ} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$





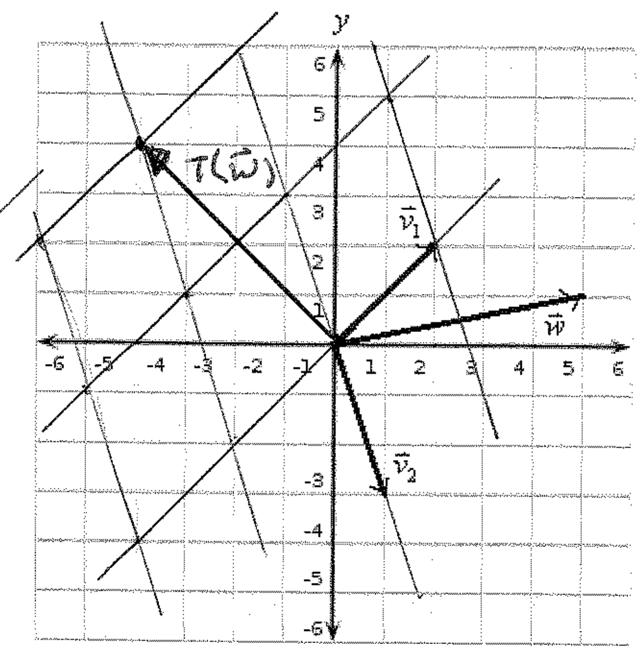
3.) (8 pts) Answer the following:

(a.) Express $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$



$\vec{w} = 2\vec{v}_1 + \vec{v}_2$

(b.) Consider a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\vec{v}_1) = 4\vec{v}_1 + 3\vec{v}_2$ and $T(\vec{v}_2) = -9\vec{v}_1 + -8\vec{v}_2$. Sketch $T(\vec{w})$



$$
\begin{aligned}
T(\vec{w}) &= T(2\vec{v}_1 + \vec{v}_2) \\
&= 2T(\vec{v}_1) + T(\vec{v}_2) \\
&= 2(4\vec{v}_1 + 3\vec{v}_2) + (-9\vec{v}_1 - 8\vec{v}_2) \\
&= -\vec{v}_1 - 2\vec{v}_2
\end{aligned}
$$

-2 if no graph

4.) (4 pts) Write $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}$ as a linear combination of the columns of A.

$$= 9\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 11\begin{bmatrix} 3 \\ 7 \end{bmatrix} + 12\begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

5.) (6 pts) Answer the following. It may help to find an example to justify your answer.

a.) True or False: The entries must be 0 or 1 in a matrix in reduced row echelon form (rref).

False! example $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

b.) True or False: If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^2$, then $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

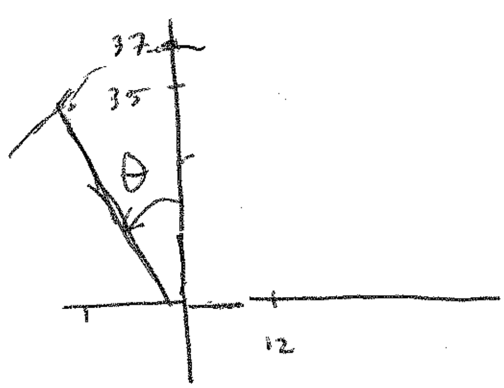
False! example $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\vec{v} = 2\vec{u}$

c.) True or False: If $A^{17} = I_2$ then the matrix A must be I_2.

False! example $A = \begin{bmatrix} \cos\left(\frac{2\pi}{17}\right) & -\sin\left(\frac{2\pi}{17}\right) \\ \sin\left(\frac{2\pi}{17}\right) & \cos\left(\frac{2\pi}{17}\right) \end{bmatrix}$

6.) (8 pts) Find a rotation matrix A that transforms $\begin{bmatrix} 0 \\ 37 \end{bmatrix}$ into $\begin{bmatrix} -12 \\ 35 \end{bmatrix}$. Give exact values.



$\tan\theta = \frac{12}{35}$

$\sin\theta = \frac{12}{37}$

$\cos\theta = \frac{35}{37}$

$A = \begin{bmatrix} 35/37 & -\frac{12}{37} \\ +\frac{12}{37} & \frac{35}{37} \end{bmatrix}$

7.) (8 pts) Prove that if A is an n x m matrix, k is a scalar, and $\vec{x} \in \mathbb{R}^m$, then $A(k\,\vec{x}) = kA\vec{x}$

Proof.

$$A(k\vec{x}) = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \left(k \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \right)$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} kx_1 \\ \vdots \\ kx_m \end{bmatrix}$$

$$= kx_1\vec{v}_1 + \dots + kx_m\vec{v}_m$$

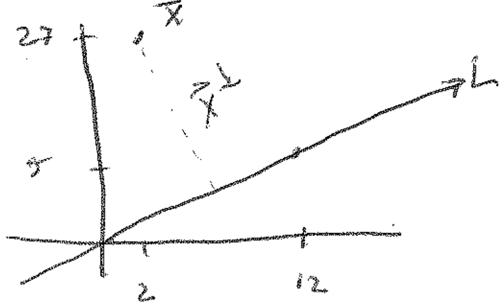
$$= k(x_1\vec{v}_1 + \dots + x_m\vec{v}_m)$$

$$= k \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \vec{x} = kA\vec{x} \quad \text{Q.E.D.}$$

8.) (8 pts) Suppose $\vec{x} = \begin{bmatrix} 2 \\ 27 \end{bmatrix}$ and L is the line $L: 12y = 5x$.

$$\vec{u} = \begin{bmatrix} 12/13 \\ 5/13 \end{bmatrix}$$

a.) Find the projection of $\vec{x}$ onto L.



$$A = \frac{1}{169} \begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix}$$

$$\vec{x}^{\parallel} = A\vec{x} = \frac{1}{169} \begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix} \begin{bmatrix} 2 \\ 27 \end{bmatrix} = \frac{1}{169} \begin{bmatrix} 1908 \\ 795 \end{bmatrix}$$

b.) Find the component of $\vec{x}$ perpendicular to the line L.

$$\vec{x}^{\perp} = \begin{bmatrix} 2 \\ 27 \end{bmatrix} - \frac{1}{169} \begin{bmatrix} 1908 \\ 795 \end{bmatrix} = \frac{1}{169} \begin{bmatrix} -1570 \\ 3768 \end{bmatrix}$$