$\overline{X} = 81.2\%$

med = 78.48

Test 3
Dusty Wilson
Math 220

Name: Key.

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert
1862 - 1943 (Prussian mathematician)

No work = no credit

Warm-ups (1 pt each)[1]: $\vec{e}_1^{\,T}\vec{e}_2 =$ 0 $\quad \vec{e}_2 \cdot \vec{e}_2 =$ 1 $\quad \vec{e}_1\vec{e}_1^{\,T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

1.) (1 pt) According to Hilbert, how much transcendent or intrinsic meaning is there in mathematics? (See above). Answer using complete English sentences.

Hilbert believed mathematics void of intrinsic/transcendent value.

2.) (10 pts) Evaluate $\det(A)$ by hand given $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{bmatrix}$. Show all steps.

$$\det(A) = -\begin{vmatrix} 3 & 2 & 4 \\ 6 & 4 & 8 \\ 6 & 4 & 12 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 4 \\ 1 & 4 & 8 \\ 2 & 4 & 12 \end{vmatrix}$$

$$= -(144 + \cancel{96} + \cancel{96} - \cancel{96} - \cancel{96} - 144) + 3(48 + \cancel{32} + 16 - \cancel{32} - 32 - 24)$$

$$= 24$$

[1] In the warm-ups, $\vec{e}_i$ refers to the standard basis vector in $\mathbb{R}^2$.

3.) (10 pts) Find the QR factorization of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \|\vec{v}_1\| = \sqrt{2}$$

$$\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad ; \quad \vec{u}_1 \cdot \vec{v}_2 = \frac{3}{\sqrt{2}}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} ; \quad \|\vec{v}_2^{\perp}\| = \frac{\sqrt{2}}{2}$$

$$\vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} ; \quad \vec{u}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \quad \text{and} \quad \vec{u}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} ; \quad \|\vec{v}_3^{\perp}\| = 2$$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{2}/2 & 1/\sqrt{2} \\ 0 & 0 & 2 \end{bmatrix} = QR.$$

4.) (10 pts) Consider the experimental observations given in the following table:

x	1	2	3	4	5	6
y	3.6	3.3	1.8	−1.8	−1.3	0.5

Find the least-squares fit of the form $y = ax + b\sin(x)$ fit to the data using techniques developed in linear algebra. ~~Give exact (fraction) values in your answer and make sure to show any formulas you use~~. 2 decimal places.

$$A = \begin{bmatrix} 1 & \sin(1) \\ 2 & \sin(2) \\ \vdots & \vdots \\ 6 & \sin(6) \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3.6 \\ 3.3 \\ 1.8 \\ -1.8 \\ -1.3 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 0.2 \\ 3.4 \end{bmatrix}$$

$$y = 0.29 + 3.4\sin(x)$$

2 pts

5.) (10 pts) Suppose V is the subspace spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$, do the following:

a.) Show $\vec{x}$ is not in V.

$$\text{rref}\left(\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 3 \\ 3 & 6 & 4 \end{bmatrix} \right) = I \quad \ldots$$

Since $\vec{v}_1, \vec{v}_2, \& \vec{x}$ are L.I., we know $\vec{x} \notin \text{span}(\vec{v}_1, \vec{v}_2)$

b.) Find $\vec{x}^{\parallel}$ (decimals are ok to 3 places)

requires Gram-Schmidt... already tested this.

c.) Find $\vec{x}^{\perp}$ (decimals are ok to 3 places)

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$$

TSJ

6.) (10 pts) Use the determinant to find out for which values of the constant λ the matrix $A - \lambda I$ fails to be invertible. Alg. mult.

Note: You may find it helpful to expand and factor by grouping. The answers are nice integers.

$$A = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

Solve $0 = \det(A - \lambda I)$

$$= \begin{vmatrix} 2-\lambda & 4 & 4 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(1-\lambda)(3-\lambda) + 1\right]$$

$$= (2-\lambda)(3 - 4\lambda + \lambda^2 + 1)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 4)$$

$$= (2-\lambda)(\lambda-2)^2$$

$\lambda = 2$ (alg. mult 3).

7.) (30 pts) True or False

a.) (True or False) All nonzero symmetric matrices are invertible.

False. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

square

b.) (True or False) If matrix A is orthogonal, then the matrix A^2 must be orthogonal as well.

A orthogonal $\Rightarrow$ $A^T A = I$

Test $(A^2)^T A^2 = A^T A^T A A = A^T I A = A^T A = I$

True

c.) (True or False) Every invertible matrix A can be expressed as the product of an orthogonal matrix and an upper triangular matrix.

True since the cols are L.I. of A.

d.) (True or False) The formula $\ker(A) = \ker(A^T A)$

True by Thm 5.4.2a

e.) (True or False) The entries of an orthogonal matrix are all less than or equal to 1.

True.

f.) (True or False) $\det(4A) = 4\det(A)$ for all 4×4 matrices A.

False. If $A = I_4$, $\det(A) = 1$ but $\det(4A) = 4^4$

g.) (True or False) If all the entries of a 7×7 matrix A are 7, then $\det(A) = 7^7$.

False. It would be zero since the rows/cols are lin. dependent.

h.) (True or False) If the determinant of a 4×4 matrix A is 4, then its rank must be 4.

True. If $\det \neq 0$, the rows are L.I.

i.) (True or False) If all the entries of a square matrix are 1 or 0, then $\det(A)$ must be 1, 0, or -1.

False. $\det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2$

j.) (True or False) If A is any symmetric matrix, then $\det(A) = 1$ or $\det(A) = -1$

False. $\det\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2.$