Test 1 – Part A **Name:** Key

Dusty Wilson

Math 220

An elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline
1908-1992 (American mathematician)

No work = no credit
No Graphing Calculators

Warm-ups (1 pt each): $\underset{2\times 1}{\begin{bmatrix}3\\4\end{bmatrix}}\underset{1\times 2}{\begin{bmatrix}1 & 2\end{bmatrix}} = \begin{bmatrix}3 & 6\\4 & 8\end{bmatrix}$ $\quad I_2 + I_2 = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ $\quad \vec{e}_1 \cdot \vec{e}_2 = 0$

1.) (1 pts) According to Kline (above), how should a good proof be written? Answer using complete English sentences.

A good proof reads like a bad poem.

2.) (8 pts) Consider $A = \begin{bmatrix}0 & 1 & 0\\3 & 3 & 4\\1 & 1 & 1\end{bmatrix}$

Find A^{-1} if it exists. If it doesn't exist, write the letters of my first name in alphabetical order.

$$\left[\begin{array}{ccc|ccc}0 & 1 & 0 & 1 & 0 & 0\\3 & 3 & 4 & 0 & 1 & 0\\1 & 1 & 1 & 0 & 0 & 1\end{array}\right] R_1 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1 & 1 & 1 & 0 & 0 & 1\\3 & 3 & 4 & 0 & 1 & 0\\0 & 1 & 0 & 1 & 0 & 0\end{array}\right] R_2 - 3R_1 \to R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1 & 1 & 1 & 0 & 0 & 1\\0 & 0 & 1 & 0 & 1 & -3\\0 & 1 & 0 & 1 & 0 & 0\end{array}\right] R_2 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1 & 1 & 1 & 0 & 0 & 1\\0 & 1 & 0 & 1 & 0 & 0\\0 & 0 & 1 & 0 & 1 & -3\end{array}\right] R_1 - R_3 \to R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1 & 1 & 0 & 0 & -1 & 4\\0 & 1 & 0 & 1 & 0 & 0\\0 & 0 & 1 & 0 & 1 & -3\end{array}\right] R_1 - R_2 \to R_1$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & -1 & 4\\0 & 1 & 0 & 1 & 0 & 0\\0 & 0 & 1 & 0 & 1 & -3\end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix}-1 & -1 & 4\\1 & 0 & 0\\0 & 1 & -3\end{bmatrix}$$

3.) (10 pts) Consider the system of linear equations:

$$\left|\begin{array}{rcrcrcrcr} & & x_2 & + & 3x_3 & + & 11x_4 & = & 17 \\ x_1 & - & 2x_2 & & & - & 3x_4 & = & -4 \\ 2x_1 & - & 3x_2 & + & x_3 & - & x_4 & = & 1 \end{array}\right|$$

a.) Write the associated coefficient matrix A

$$\begin{bmatrix} 0 & 1 & 3 & 11 \\ 1 & -2 & 0 & -3 \\ 2 & -3 & 1 & -1 \end{bmatrix}$$

b.) Solve the system using Gauss-Jordan Elimination. Express your solution as a vector. Fractions may be required …

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & 11 & 17 \\ 1 & -2 & 0 & -3 & -4 \\ 2 & -3 & 1 & -1 & 1 \end{array}\right] R_1 \Leftrightarrow R_2 \quad \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -4 \\ 0 & 1 & 3 & 11 & 17 \\ 0 & 0 & 1 & 3 & \cancel{5}4 \end{array}\right] R_2 - 3R_3 \to R_2$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -4 \\ 0 & 1 & 3 & 11 & 17 \\ 2 & -3 & 1 & -1 & 1 \end{array}\right] R_3 - 2R_1 \to R_3 \quad \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right] R_1 + 2R_2 \to R_1$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -4 \\ 0 & 1 & 3 & 11 & 17 \\ 0 & 1 & 1 & 5 & 9 \end{array}\right] R_3 - R_2 \to R_3 \quad \Rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right] \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & -3 & -4 \\ 0 & 1 & 3 & 11 & 17 \\ 0 & 0 & -2 & -6 & -8 \end{bmatrix} -\tfrac{1}{2}R_3 \to R_3 \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

c.) What is the rank of the coefficient matrix A found in (a.)?

rank$(A) = 3$

Test 1 – Part B **Name:** Key

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Math 220

No work = no credit

1.) (8 pts) Use linear algebra to find the polynomial of degree 2 (a polynomial of the form $f(t)=a+bt+ct^2$) whose graph goes through the points $(-1,1)$, $(2,3)$, and $(3,13)$.

$(-1,1): \; a - b + c = 1$

$(2,3): \; a + 2b + 4c = 3$

$(3,13): \; a + 3b + 9c = 13$

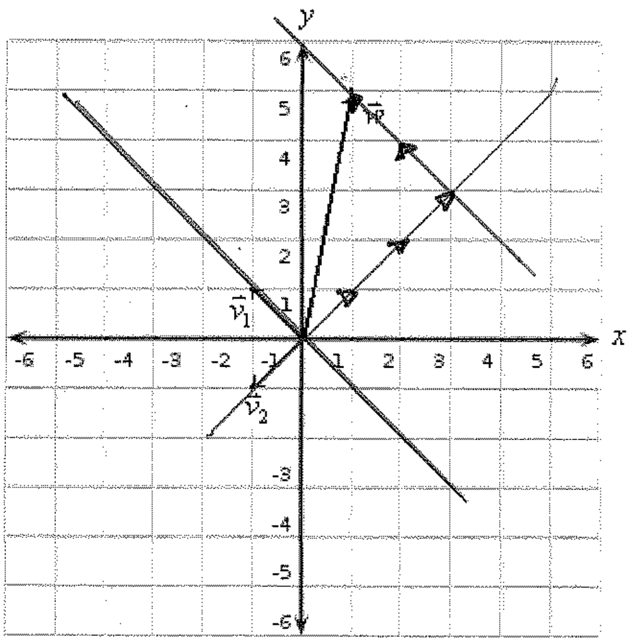
$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 13 \end{bmatrix}$$

$$\text{so } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ -5/3 \\ 7/3 \end{bmatrix} \quad \text{and} \quad f(t) = -3 - \frac{5}{3}t + \frac{7}{3}t^2$$

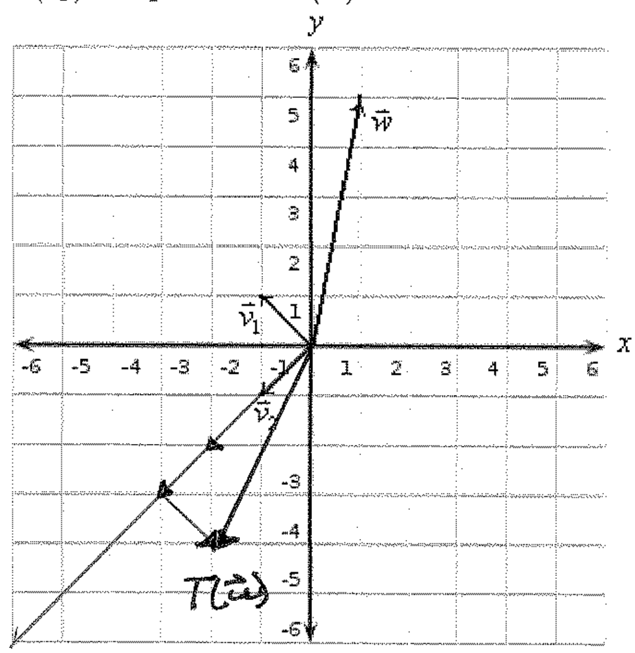
2.) (8 pts) Answer the following:

(a.) Express $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$\vec{w} = 2\vec{v}_1 - 3\vec{v}_2$



(b.) Consider a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\vec{v}_1) = -\frac{1}{2}\vec{v}_1$ and $T(\vec{v}_2) = -\vec{v}_2$. Sketch $T(\vec{w})$ on the same axes.



$$\begin{aligned} T(\vec{w}) &= T(2\vec{v}_1 - 3\vec{v}_2) \\ &= 2T(\vec{v}_1) - 3T(\vec{v}_2) \\ &= 2\cdot -\tfrac{1}{2}\vec{v}_1 - 3\cdot(-v_2) \\ &= -\vec{v}_1 + 3\vec{v}_2 \end{aligned}$$

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3.) (6 pts) Answer the following. It may help to find an example to justify your answer.

a.) True or False: If matrices A_{2x2} and B_{2x2} are invertible, then the matrix $A+B$ is invertible.

False: $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

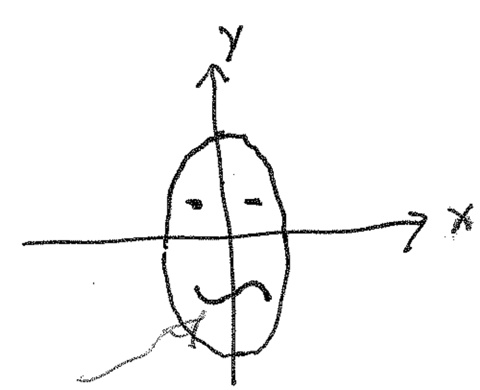
b.) True or False: If $A^{17} = I_2$ then matrix A must be I_2.

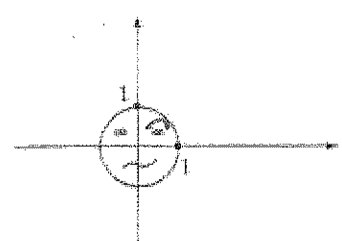
False : A could rotate by $\theta = \frac{2\pi}{17}$

c.) True or False: There exists an invertible $n \times n$ matrix with two identical rows.

False: rref of such an A would show rank < N.

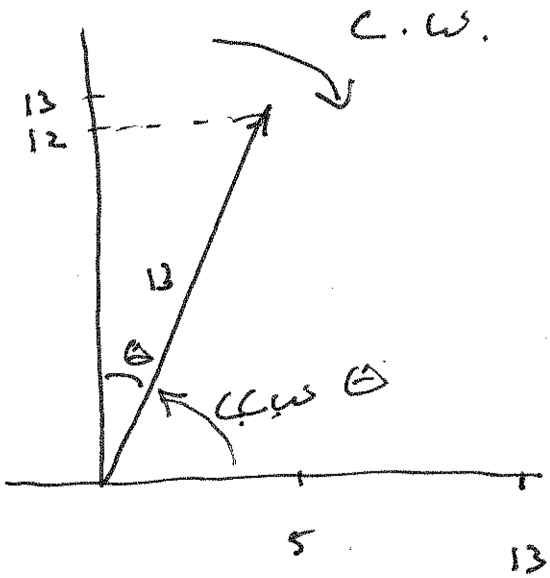
4.) (8 pts) Consider the circular face. For the matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ draw a sketch showing the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ on this face.





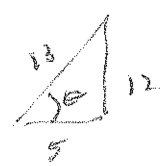
2 pts

5.) (8 pts) Find a rotation matrix A that transforms $\begin{bmatrix}13\\0\end{bmatrix}$ into $\begin{bmatrix}5\\12\end{bmatrix}$.



$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\frac{12}{13} & -\frac{5}{13}\\ \frac{5}{13} & \frac{12}{13}\end{bmatrix}$$



$$A = \begin{bmatrix}\frac{5}{13} & -\frac{12}{13}\\ \frac{12}{13} & \frac{5}{13}\end{bmatrix}$$

$\cos\theta = \frac{12}{13}$; $\sin\theta = \frac{5}{13}$

6.) (8 pts) Prove that if $T:\mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then $T(\vec{v}+\vec{w}) = T(\vec{v})+T(\vec{w})$ for all $\vec{v},\vec{w}\in\mathbb{R}^m$

Proof.

Let the linear trans $T:\mathbb{R}^m \mapsto \mathbb{R}^n$ be given and take $\vec{v},\vec{\omega}\in\mathbb{R}$.

$\Rightarrow \exists\, A_{n\times m}$ s.t. $T(\vec{v}) = A\vec{v}$

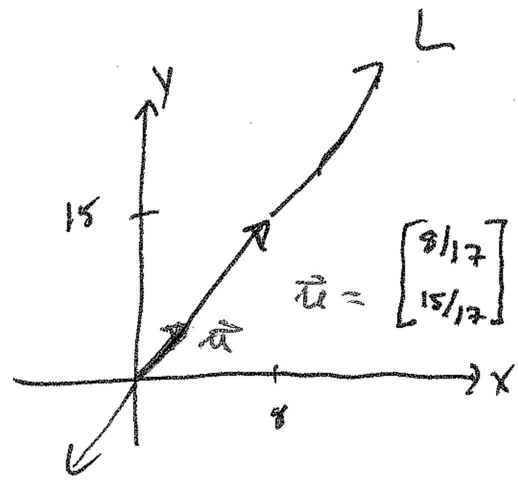
$$\Rightarrow T(\vec{v}+\vec{\omega}) = A(\vec{v}+\vec{\omega})$$
$$= A\vec{v} + A\vec{\omega}$$
$$= T(\vec{v}) + T(\vec{\omega})$$

Q.E.D.

7.) (8 pts) Write $c_1\vec{a}_1+\dots+c_m\vec{a}_m$ as the product of a matrix A and a vector if $c_1,\dots,c_m \in \mathbb{R}$ and $\vec{a}_1,\dots,\vec{a}_m \in \mathbb{R}^n$. What are the dimensions of A?

$$c_1 a_1 + \dots + c_m a_m = \underbrace{\overset{A}{\begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_m \\ | & & | \end{bmatrix}}}_{n \times m} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

A is $n \times m$.



8.) (8 pts) Suppose $\vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and L is the line $L: 8y = 15x$.

a.) Find the projection of $\vec{x}$ onto L.

$$\vec{x}^{\parallel} = (\vec{x} \cdot \vec{u})\,\vec{u}$$

$$= \underbrace{\left(\begin{bmatrix} -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8/17 \\ 15/17 \end{bmatrix} \right)}_{\frac{67}{17}} \begin{bmatrix} 8/17 \\ 15/17 \end{bmatrix}$$

$$= \frac{67}{289} \begin{bmatrix} 8 \\ 15 \end{bmatrix} = \begin{bmatrix} 536/289 \\ 1005/289 \end{bmatrix}$$

b.) Find the component of $\vec{x}$ perpendicular to the line L.

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix} - \frac{67}{289} \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} -825/289 \\ 440/289 \end{bmatrix}$$