Test 2
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Math 220

Name: KEY

... we can repudiate completely and which we can abandon without regret because one does not know what this pretended sign signifies nor what sense one ought to attribute to it..

Augustin Cauchy
1789 – 1857 (French mathematician)

No work = no credit

$\vec{e}_2 \cdot \vec{e}_1 = 0$ $\quad$ $\vec{e}_1^{\,T} \vec{e}_1 = [1]$ $\quad$ $\vec{e}_2 \vec{e}_2^{\,T} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Warm-ups (1 pt each): $\cancel{\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}}=$____ $\quad$ $\cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix}}=$____ $\quad$ $\cancel{A^{13}\left(A^{-1}\right)^{6}}=$____

1.) (1 pt) The quote above gives Cauchy's understanding of $\sqrt{-1}$. Paraphrase Cauchy's sentiments (see above). Answer using complete English sentences.

There is no reason to keep imaginary numbers.

2.) (10 pts) If $\vec{x} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$, the basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$, and $[\vec{y}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $[\vec{x}]_B$ and $\vec{y}$.

$$[\vec{y}]_B = 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 25 \end{bmatrix} = \vec{y}$$

$$\vec{x} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \end{array}\right]\right) \Rightarrow [\vec{x}]_B = \begin{bmatrix} 21 \\ -22 \\ 8 \end{bmatrix}$$

3.) (10 pts) Consider the matrix $A = \begin{bmatrix} 4 & 8 & 1 & 1 & 7 \\ 3 & 6 & 1 & 2 & 8 \\ 2 & 4 & 1 & 9 & 21 \\ 1 & 2 & 3 & 2 & 8 \end{bmatrix}$

a.) Find the ~~image~~ basis of the image of A.

basis

$$\text{Im}(A) = \text{Span}\left(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix} \right)$$

Note: $\text{rref}\left(\left[A \middle| \vec{0}\right]\right) = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$x_1 = -2x_2 - x_5$

x_2

$x_3 = -x_5$

$x_4 = -2x_5$

x_5

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

b.) Find the kernel of A.

$$\text{Ker}(A) = \text{Span}\left(\begin{bmatrix} -2 \\ +1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \right)$$

1 pt.

c.) rank(A) = 3 and nullity(A) = 2

4.) (6 pts) Describe a basis, the conditions a basis must satisfy, and the meaning of those conditions.

A basis is a set of vectors that spans the space & is L.I.

L.I.: Vectors are L.I. if no non-trivial lin. comb. adds to $\vec{0}$.

Span: vectors span a space if every vector in the space can be represented as a lin. comb. of the spanning set.

5.) (5 pts) Consider a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$. Prove that the kernel is closed under addition.

Let $\vec{x}, \vec{y} \in \mathbb{R}^m$ be in $\ker(T)$

consider $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

$= \vec{0} + \vec{0}$

$= \vec{0}$

Hence $\vec{x}+\vec{y} \in \ker(T)$

6.) (4 pts) Find a basis for the space of all upper-triangular 2x2 matrices and determine its dimension.

Basis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and dimension: 3

7.) (6 pts) Show that the transformation $T(f(t)) = f(3)$ from P^2 to $\cancel{\mathbb{R}^2}$ $\mathbb{R}$ is linear and determine if the transformation is an isomorphism (justify your answer).

Let $f, g \in P^2$ and $k \in \mathbb{R}$.

$\Rightarrow f(t) = at^2 + bt + c$; $g(t) = dt^2 + et + h$

(1) $T(f+g) = (a+d)\cdot 3^2 + (b+e)\cdot 3 + (c+h)$

$= (a\cdot 3^2 + b\cdot 3 + c) + (d\cdot 3^2 + e\cdot 3 + h)$

$= T(f) + T(g)$

(2) $T(kf) = ka\cdot 3^2 + kb\cdot 3 + kc$

$= k(a\cdot 3^2 + b\cdot 3 + c)$

$= k\,T(f)$.

Hence T is a lin transformation.

T is <u>not</u> an isomorphism. If $f = 0$ & $g = 3-t$, $T(f) = T(g)$ but $f \neq g$.

8.) (6 pts) Find the matrix of the linear transformation $T(a+bi) = b$ and bases for the image and kernel of T.

$$\begin{tikzcd}
a+bi \arrow[r, "T"] \arrow[d] & b \\
\begin{bmatrix} a \\ b \end{bmatrix} \arrow[r, "B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}"] & \begin{bmatrix} b \\ 0 \end{bmatrix} \arrow[u]
\end{tikzcd}$$

image(A) = span $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

ker(A) = span $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$