7.3: Eigenvectors.

ex1: recall that the eigenvalue for $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

is 2 (alg. mult. 2).

To find the corresponding eigenvector(s), we look for nontrivial soln. to

$$\underbrace{(A - 2I)}\vec{v} = \vec{0}$$

$$\begin{array}{cc} 1 & -1 \\ \end{array}$$
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{v} = \vec{0} \quad \text{whose soln. are } \vec{v} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for $a \neq 0$

DFN: Consider an eigenvalue λ of an $n \times n$ matrix A. Then the $\ker(A - \lambda I_n)$ is called the eigenspace associated w/ λ and denoted by E_λ:

$$E_\lambda = \ker(A - \lambda I_n) = \{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda\vec{v} \}$$

ex1 rev: $E_2 = \text{span}(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and we say $\lambda = 2$ has geometric multiplicity 1.

Note: all vectors in E_2 are eigenvectors except $\vec{v} = \vec{0}$.

DFN: Consider an eigenvalue λ of an $n \times n$ matrix A. The $\dim(E_\lambda)$ is called the geometric mult. of the eigenvalue λ. The geo. mult = nullity of $A - \lambda I_n$.

ex2: recall that the eigenvals for $B = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix}$

are $\lambda = 0$ & $\lambda = -1$ (alg. mult. 2). Find the eigenspaces.

$\lambda = 0 : E_0 = \ker(A - 0I) = \ker(A) = \operatorname{span}\left(\begin{bmatrix} -1/2 \\ 1 \\ -1 \end{bmatrix} \right)$

w/ geometric mult. 1.

$$\lambda = -1 : E_{-1} = \ker(A + 1I) = \ker\left(\begin{array}{c} \begin{matrix} 1 & -1 & 2 \end{matrix} \\ \begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & 1 \\ -2 & -2 & 0 \end{bmatrix} \end{array} \right)$$

$$= \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right)$$

w/ geometric mult. 1.

Dfn: Consider an $n \times n$ matrix A. A basis of $\mathbb{R}^n$ consisting of eigenvectors of A is called an eigenbasis for A.

ex: The roadrunner & coyote example.

ex: orthogonal projection onto a plane.

Thm: If an $n \times n$ matrix A has n distinct eigenvals, then there exists an eigenbasis for A.

Q: what if the eigenvals aren't distinct.

Thm: Suppose matrix A is similar to B. Then

(a) matrices A & B have the same characteristic poly.

□ proof

If $B = S^{-1}AS$, then

$$\begin{aligned} \det(B - \lambda I) &= \det(S^{-1}AS - \lambda I) \\ &= \det(S^{-1}AS - S^{-1} \cdot \lambda I\, S) \\ &= \det(S^{-1}(A - \lambda I)S) \\ &= \det(S^{-1}) \det(A - \lambda I) \det(S) \\ &= \det(A - \lambda I) \ \blacksquare \end{aligned}$$

(b) matrices A & B have the same eigenvals, and the same determinant.

Thm: If λ is an eigenval. of a square matrix A, then (geo mult of λ) $\leq$ (alg mult of λ)