

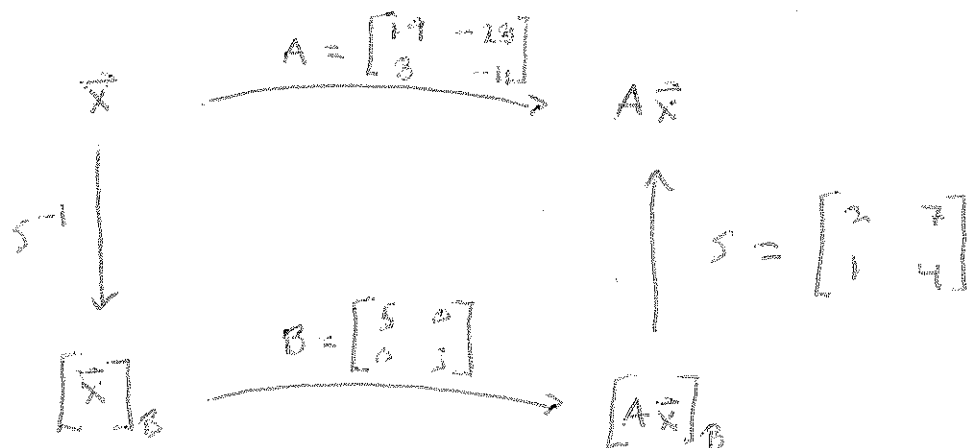
## 7.1: Diagonalization

ex: Consider a linear transformation  $T$

s.t.  $T(\vec{v}_1) = 5\vec{v}_1$ ,  $T(\vec{v}_2) = 3\vec{v}_2$  w/  $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  &  $\vec{v}_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

(a) Find the  $S$  &  $B$  matrices

(b) calculate  $A = S B S^{-1}$



We say  $T(\vec{x}) = A\vec{x}$  is diagonalizable since  $A$  is similar to a diagonal matrix  $B$ .

Diagonal matrices are our friends because they simplify calculations by focusing on the most important relationships.

ex:  $A^5$  vs  $B^5$   
 $A^5$  vs  $(S B S^{-1})^5$

ex/rev: Notice that  $A\vec{v}_1 = 5\vec{v}_1$  and  $A\vec{v}_2 = 3\vec{v}_2$ .

We call 5 & 3 eigenvalues of  $A$  w/ associated eigenvectors  $\vec{v}_1$  &  $\vec{v}_2$ .

Def: A non-zero vector  $\vec{v}$  is called an eigenvector if  $A\vec{v} = \lambda\vec{v}$  for a scalar  $\lambda$ ,  $\lambda$  is an eigenvalue associated w/  $\vec{v}$ .

ex: verify  $A = \begin{bmatrix} 15 & -35 \\ 6 & -14 \end{bmatrix}$  has eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}.$$

Def: A basis  $\vec{v}_1, \dots, \vec{v}_n$  for  $\mathbb{R}^n$  is called a eigenbasis for  $A_{n \times n}$  if  $A\vec{v}_i = \lambda_i\vec{v}_i$  for  $i=1, \dots, n$ .

Thm:  $A_{n \times n}$  is diagonalizable iff  $\exists$  an eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  w/ associated eigenvalues  $\lambda_1, \dots, \lambda_n$  for  $A$ .

Then

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \text{ and } B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

ex: Diagonalize  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$  w/ one

7.1  
3/2

eigen vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  & eigenvalue  $\lambda = 1$ .

FAIL  
SHEAR

ex: Rotation  $A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

FAIL  
Rotation

ex: projection  $A = \frac{1}{119} \begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix}$

Find eigenvectors & values using geometry.

#### Various characterizations of invertible matrices

For an  $n \times n$  matrix  $A$ , the following statements are equivalent.

- i.  $A$  is invertible.
- ii. The linear system  $A\vec{x} = \vec{b}$  has a unique solution  $\vec{x}$ , for all  $\vec{b}$  in  $\mathbb{R}^n$ .
- iii.  $\text{ref } A = I_n$ .
- iv.  $\text{rank } A = n$ .
- v.  $\text{im } A = \mathbb{R}^n$ .
- vi.  $\text{ker } A = \{\vec{0}\}$ .
- vii. The column vectors of  $A$  form a basis of  $\mathbb{R}^n$ .
- viii. The column vectors of  $A$  span  $\mathbb{R}^n$ .
- ix. The column vectors of  $A$  are linearly independent.
- x.  $\det A \neq 0$ .
- xi.  $0$  fails to be an eigenvalue of  $A$ .

Eigen vectors are cool, but we are left w/ two logical questions.

- ① How do we find  $\lambda$
- ② How do we find eigenvectors