7.1 : Diagonalization

ex: Consider a linear transformation T
s.t. $T(\vec{v}_1) = 5\vec{v}_1$, $T(\vec{v}_2) = 3\vec{v}_2$ w/ $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

(a) Find the S & B matrices

(b) Calculate $A = SBS^{-1}$

$$
\begin{tikzcd}
\vec{x} \arrow[rr, "A = \begin{bmatrix} 19 & -28 \\ 8 & -11 \end{bmatrix}"] \arrow[d, "S^{-1}"'] & & A\vec{x} \\
{[\vec{x}]_B} \arrow[rr, "B = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}"] & & {[A\vec{x}]_B} \arrow[u, "S = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}"']
\end{tikzcd}
$$

We say $T(\vec{x}) = A\vec{x}$ is diagonalizable since A is similar to a diagonal matrix B.

Diagonal matrices are our friends because they simplify calculations by focusing on the most important relationships.

ex: A^5 vs. B^5

A^5 vs $(SBS^{-1})^5$

ex 1 rev: Notice that $A\vec{v}_1 = 5\vec{v}_1$ and $Av_2 = 3\vec{v}_2$.
We call 5 & 3 eigenvalues of A w/ associated eigenvectors $\vec{v}_1$ & $\vec{v}_2$

Dfn: A non-zero vector $\vec{v}$ is called an eigenvector if $A\vec{v} = \lambda\vec{v}$ for a scalar λ. λ is an eigenvalue associated w/ $\vec{v}$.

ex: verify $A = \begin{bmatrix} 15 & -35 \\ 6 & -14 \end{bmatrix}$ has eigenvectors
$\vec{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

Dfn: A basis $\vec{v}_1, \ldots, \vec{v}_n$ for $\mathbb{R}^n$ is called a eigenbasis for $A_{n\times n}$ if $A\vec{v}_i = \lambda_i\vec{v}_i$ for $i=1,\ldots,n$.

Thm: $A_{n\times n}$ is diagonalizable iff $\exists$ an eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$ w/ associated eigenvalues $\lambda_1, \ldots, \lambda_n$ for A.
Then

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

ex: Diagonalize $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ w/ one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & eigenvalue $\lambda = 1$.

$\begin{bmatrix} 7/1 \\ 3/3 \end{bmatrix}$

FAIL SHEAR

ex: Rotation $A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ \frac{1}{2} & \sqrt{3}/2 \end{bmatrix}$

FAIL Rotation

ex: projection $A = \frac{1}{169}\begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix}$

Find eigenvectors & values using geometry.

Various characterizations of invertible matrices

For an $n \times n$ matrix A, the following statements are equivalent.

- **i.** A is invertible.
- **ii.** The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ in $\mathbb{R}^n$.
- **iii.** rref $A = I_n$.
- **iv.** rank $A = n$.
- **v.** im $A = \mathbb{R}^n$.
- **vi.** ker $A = \{\vec{0}\}$.
- **vii.** The column vectors of A form a basis of $\mathbb{R}^n$.
- **viii.** The column vectors of A span $\mathbb{R}^n$.
- **ix.** The column vectors of A are linearly independent.
- **x.** det $A \neq 0$.
- **xi.** 0 fails to be an eigenvalue of A.

Eigen vectors are cool, but we are left w/ two logical questions.

1. How do we find λ
2. How do we find eigenvectors