4.3: Matrix of a L.T.

ex: $T(f(t)) = f(2t-1)$ from P_2 to P_2

$$\left.\begin{aligned} T(1) &= 1 \\ T(t) &= 2t-1 \\ T(t^2) &= (2t-1)^2 = 4t^2-4t+1 \end{aligned}\right\} \text{L.I.}$$

T is an isomorphism. $\big(\operatorname{im}(T) = P_2$ & $\ker(T) = 0\big)$

L_u is called the coordinate transformation from P_2 to $\mathbb{R}^3$

$$\begin{tikzcd}
a+bt+ct^2 \arrow[r, "T"] \arrow[d, "L_u"'] & a+b(2t-1)+c(4t^2-4t+1) = (a-b+c)+(2b-4c)t+4ct^2 \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} \arrow[r, "B"] & \begin{bmatrix} a-b+c \\ 2b-4c \\ 4c \end{bmatrix} \arrow[u, "L_u^{-1}"']
\end{tikzcd}$$

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\underbrace{[T(1)]_u \quad [T(t)]_u \quad [T(t^2)]_u}_{\text{must be in the same order as } [\vec{x}]_B.}$$

ex: $T(z) = (p+iq)z$ from $\mathbb{C}$ to $\mathbb{C}$, p & $q \in \mathbb{R}$.

$$T(1) = (p+iq)1 = p+iq$$
$$T(i) = (p+iq)i = -q+pi$$

L.I.

T is an isomorphism. $\left(im(T) = \mathbb{C} \text{ and } ker(T) = 0 \right)$

$L_{\mathcal{U}}$ is the coordinate transformation from $\mathbb{C}$ to $\mathbb{R}^2$.

$$\begin{tikzcd}
z = a+bi \arrow[rr, "(p+iq)z", "T"'] \arrow[d, "L_{\mathcal{U}}"] & & (ap-bq)+i(aq+bp) \\
\left[z\right]_{\mathcal{U}} = \begin{bmatrix} a \\ b \end{bmatrix} \arrow[rr, "B"] & & \begin{bmatrix} ap-bq \\ aq+bp \end{bmatrix} = \left[T(z)\right]_{\mathcal{U}} \arrow[u, "L_{\mathcal{U}}^{-1}"']
\end{tikzcd}$$

$$\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \left[T(1)\right]_{\mathcal{U}} & \left[T(i)\right]_{\mathcal{U}} \end{array}$$

ex: $T(m) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} m$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$,

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

only two L.I. matrices so T is not an isomorphism

$$\operatorname{im}(T) = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right)$$

$$\ker(T) = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right)$$

L_u is a coordinate transformation from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^4$.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{tikzcd}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \arrow[r] \arrow[d, "L_u"] & \begin{bmatrix} a+c & b+d \\ 2a+2c & 2b+2d \end{bmatrix} \arrow[d, "L_u"] \\
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_u \arrow[r] & \begin{bmatrix} a+c \\ b+d \\ 2a+2c \\ 2b+2d \end{bmatrix}_u
\end{tikzcd}$$

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$