4.2 : Lin Trans and Isomorphisms

<u>Def</u>: Consider two lin. spaces V & W. $T: V \mapsto W$ is a lin. trans. if

(a) $T(f+g) = T(f) + T(g)$

(b) $T(kf) = kT(f)$

$\forall f, g \in V$ and scalars k.

(i) $\text{im}(T) = \{T(f) \mid f \in V\}$

(ii) $\ker(T) = \{f \in V \mid T(f) = 0\}$

If the image of T is finite dim. then

$\dim(\text{im}\, T) = \text{rank}(T)$

If the ker of T is finite dim. then

$\dim(\ker T) = \text{nullity}(T)$

And if V is finite dim:

$\dim(V) = \text{rank}(T) + \text{nullity}(T)$

$= \dim(\text{im}\, T) + \dim(\ker T)$

<u>ex1</u>: Is the transformation linear?

$T(m) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} m$ from $\mathbb{R}^{2\times 2} \mapsto \mathbb{R}^{2\times 2}$

<u>ex2</u>: Is the transformation linear?

$T(m) = PmQ$ where $P = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ & $Q = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$

from $\mathbb{R}^{2\times 2} \mapsto \mathbb{R}^{2\times 2}$

ex1: Consider $T: \mathbb{R}^{2\times 2} \mapsto \mathbb{R}^{2\times 2}$ where

$$T(m) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} m$$

(a) is T linear?

Let $A, B \in \mathbb{R}^{2\times 2}$ and $k \in \mathbb{R}$ be given.

(i) $T(A+B) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}(A+B) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}A + \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}B$

$= T(A) + T(B)$

(ii) $T(kA) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}(kA) = k\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}A = kT(A)$.

Hence T is linear.

(b) Find the $\ker(T)$.

solve $T\left(\overset{m}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\right) = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+6c & 3a+6c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a+2c=0$ and $b+2d=0$

so $m = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$ $\longleftarrow$ basis: $\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$

and $\ker(T) = \text{span}\left(\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}\right)$

(c) Find $\text{im}(T)$.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are LI and not in the kernel,

$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

$\dim(\text{Im}(T)) = \dim(\mathbb{R}^{2\times 2}) - \dim(\ker(T)) = 2$

since I have 2 LI matrices in the image they form a basis for the image.

(d) T is not an isomorphism since $(T) \neq \{0\}$

ex2: Consider $T: P_2 \longmapsto P_2$ where $T(f(t)) = f''(t) + 4f'(t)$

(a) Is T linear?

Let $g(t) = at^2 + bt + c$ and $h(t) = dt^2 + et + f$ be given w/ scalar $k \in \mathbb{R}$

(i) $$T(g+h) = T\left((a+d)t^2 + (b+e)t + (c+f)\right)$$
$$= 2(a+d) + 4\left[2(a+d)t + (b+e)\right]$$
$$= \underbrace{2a + 4\left[2at + b\right]}_{T(g(t))} + \underbrace{2d + 4\left[2dt + e\right]}_{T(h(t))}$$

(ii) $$T(kg) = T(kat^2 + kbt + kc)$$
$$= 2ka + 4\left[2kat + kb\right]$$
$$= k\left(2a + 4\left[2at + b\right]\right)$$
$$= k\,T(g)$$ Hence T is linear.

(b) Find $\ker(T)$.

From above we see that T sends the constant c to zero.
Hence $\ker(T) = \text{span}(1)$

(c) Find $\text{im}(T)$.

From (a.i) we see $T(at^2 + bt + c) = 2a(1+4t) + 4b$,
so $\text{im}(T) = \text{span}(2 + 8t, 4) = \text{span}(1 + 4t, 1)$

(d) Since $\ker(T) \neq \{0\}$, T is not an isomorphism.

Find the kernel & image in (ex1) & (ex2).

<u>Dfn</u>: An <u>invertible linear transformation</u> is called an isomorphism.

Coordinate Transformations are isomorphisms.

<u>Key</u>: Any n-dim. lin. space is isomorphic w/ $\mathbb{R}^n$.

nice example: P_n is isomorphic w/ $\mathbb{R}^{n+1}$.

<u>Thm</u>: Properties of isomorphisms.

- $T: V \mapsto W$ is an isomorphism iff $\ker(T) = \{0\}$ and $\text{im}(T) = W$.
- If V is isomorphic to W, then $\dim V = \dim W$
- If $T: V \mapsto W$ is a L.T. w/ $\ker(T) = \{0\}$ then T is an isomorphism.
- If $T: V \mapsto W$ is a L.T. w/ $\text{im}(T) = W$. If $\dim V = \dim W$ then T is an isomorphism.

Are (ex1) & (ex2) isomorphisms?

Carefully study flowchart in 4.2

ex 3: $T(f(t)) = f(7)$ from P_2 to $\mathbb{R}$.

LT

ker

im

isomorphism

ex 4: $T(f(t)) = f'(t)$ from P_2 to P_2

LT

ker

im

isomorphism