4.1: Intro to Linear Spaces

Dfn: V is a linear space (vector spaces) if $\forall\ f, g \in V$ and $c, k \in \mathbb{R}$

put this up on doc cam and ask students to copy into notes

(1) $(f+g)+h = f+(g+h)$

(2) $f+g = g+f$

(3) $\exists$ a neutral element $n \in V$ s.t. $f+n = f\ \forall\ f \in V$.

n is unique and denoted by 0

(4) $\forall\ f \in V\ \exists\ g \in V$ s.t. $f+g=0$. This g is unique & denoted by $(-f)$

(5) $k(f+g) = kf + kg$

(6) $(c+k)f = cf + kf$

(7) $c(kf) = (ck)f$

(8) $1f = f$.

Key: A linear space is a set over which addition and scalar mult. are defined. Zero must be defined.

examples $\mathbb{R}^n$

$F(\mathbb{R}, \mathbb{R})$ (set of fcts from $\mathbb{R}$ to $\mathbb{R}$)

$\mathbb{R}^{n \times m}$ matrices.

$\mathbb{R}^{\infty}$ The set of all infinite sequences of real numbers where addition & scalar mult is defined term by term

$\mathbb{C}$

Dfn: A subset W of a linear space V is called a subspace of V if.

(a) W contains the neutral element 0 of V.

(b) W is closed under addition.

(c) W is closed under scalar mult.

see graph/flowchart in 4.1

Which are subspaces.

Ex: Diagonal 3×3 matrices (yes)

Ex: 3×3 matrices w/ non-neg. entries (no)

Ex: 3×3 matrices in RREF form. (no)

Ex: The geometric sequences. (no)

Dfn: ① Span, ② LI, ③ Basis, & ④ coords in a linear space.
Consider $f_1, \dots, f_n$ in a linear space V.

① SPAN: $f_1, \dots, f_n$ span V if every $f \in V$ is a lin. comb of $f_1, \dots, f_n$.

② LI: $f_1, \dots, f_n$ are LI iff $c_1 f_1 + \dots + c_n f_n = 0$ has only the trivial soln

③ Basis: $f_1, \dots, f_n$ are a basis of V if they are LI and span V.

④ coordinates: If $f_1, \dots, f_n$ are a basis for V, then every $f \in V$ can be written uniquely as $c_1 f_1 + \dots + c_n f_n = f$ and we call $c_1, \dots, c_n$ the coordinates of f.

Thm: If a lin. space V has a basis w/ n elements, then all other bases of V consist of n elements as well. We say $\dim(V) = n$.

To find a basis...

(a) write a typical element in terms of arb. consts.

(b) Using the arb. constants as coefficients, express your typical element as a lin. comb. of some elements of V. (SPAN)

(c) verify that these elements of V are L.I.

... if so, it is a basis.

Find a basis & determine the dimension.

<u>ex</u>: The space of all diagonal 2×2 matrices.

basis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

dim: 2

<u>ex</u>: The space of all polys $f(t)$ in P_2 s.t. $f(1)=0$

Let $f(t) = at^2 + bt + c$

$f(1) = a + b + c = 0 \Rightarrow c = -a - b$

$\Rightarrow f(t) = at^2 + bt - a - b$

$= a(t^2 - 1) + b(t - 1)$

basis: $t^2 - 1$ and $t - 1$

dim: 2

<u>ex</u>: The space of all matrices S s.t. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left.\begin{aligned} LHS &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \\ RHS &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \end{aligned}\right\}$$

since LHS = RHS we have $c = a$; $d = -b$; $a = c$; $b = -d$

or: $S = \begin{bmatrix} a & b \\ a & -b \end{bmatrix}$

basis: $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

dim: 2.