3.4: coordinates

ex! (a) The vector $\begin{bmatrix}1\\2\\3\end{bmatrix} = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$

This is of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ where

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a basis of the subspace.

we call $\vec{x} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ the coordinate vector

(b) Another basis for the subspace $\mathbb{R}$ is: $\left\{\begin{bmatrix}0\\4\\2\end{bmatrix}, \begin{bmatrix}-2\\+1\\0\end{bmatrix}, \begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$

Notice that $\begin{bmatrix}1\\2\\3\end{bmatrix} = 1\begin{bmatrix}0\\4\\2\end{bmatrix} - 1\begin{bmatrix}-2\\1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-2\\-2\\2\end{bmatrix}$

$= 1\vec{v}_1 - 1\vec{v}_2 + \frac{1}{2}\vec{v}_3$

So $\begin{bmatrix}1\\-1\\1/2\end{bmatrix}_{\mathcal{B}}$ is the coordinate vector w/ respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Thus since $\vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ w/ $\vec{v}_i$ as above

we write $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}1\\-1\\1/2\end{bmatrix}_{\mathcal{B}}$

DFN: Consider a basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of a subspace V of $\mathbb{R}^n$. $\vec{x} \in V$ can be written as

$$\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$$

and the $\mathcal{B}$-coord vec of $\vec{x}$ is $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\\vdots\\c_m\end{bmatrix}_{\mathcal{B}}$

and if $S = \begin{bmatrix}| & & |\\\vec{v}_1 & \dots & \vec{v}_m\\| & & |\end{bmatrix}$, then $S[\vec{x}]_{\mathcal{B}} = \vec{x}$.

Thm: If $\mathcal{B}$ is a basis of a subspace V of $\mathbb{R}^n$, then

(a) $[\vec{x}+\vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ for all $\vec{x},\vec{y} \in V$

(b) $[k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}}$ and scalars k.

□ proof of (a.).

Let $\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ and $\vec{y} = d_1\vec{v}_1 + \dots + d_m\vec{v}_m$

where $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ is a basis for V

$$\Rightarrow [\vec{x}+\vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_m+d_m \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

ex2: Consider the basis $\mathcal{B}$ of $\mathbb{R}^3$ consisting of vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$; $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$; $\vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$

(a) if $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, find $[\vec{x}]_{\mathcal{B}}$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 4 & 8 & 0 \end{array}\right] \right) \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 8 \\ -12 \\ 5 \end{bmatrix}_{\mathcal{B}}$$

(b) if $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}}$, find $\vec{y}$

$$\Rightarrow S = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 8 \end{bmatrix} \text{ and } S[\vec{y}]_{\mathcal{B}} = \vec{y} = \begin{bmatrix} 6 \\ 20 \\ 33 \end{bmatrix}$$

Snow Boarding Analogy.

outsider view

NS EW + alt.

$$\begin{tikzcd} \vec{x} \arrow[d, bend right, "S^{-1}"'] \\ {[\vec{x}]_B} \arrow[u, bend right, "S"'] \end{tikzcd}$$

your pov

Paused world... no motion... just changing perspectives.

outsider's view of motion.

outsider view

$$\begin{tikzcd}[column sep=huge, row sep=large] \vec{x} \arrow[r, bend left, "T"] \arrow[d, "S^{-1}"'] & T(\vec{x}) \\ {[\vec{x}]_B} \arrow[r, "B"'] & {[T(\vec{x})]_B} \arrow[u, "S"'] \end{tikzcd}$$

Path #1

Path #2

Before

After

outsider view

your pov

your view of motion

your pov

Finding the B matrix given T and $B = \{\vec{v}_1, \dots, \vec{v}_n\}$

$$B = \begin{bmatrix} | & & | \\ T(\vec{v}_1) & \cdots & T(\vec{v}_n) \\ | & & | \end{bmatrix}$$

so $B = \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix}$

ex: If $T(c_1\vec{v}_1 + c_2\vec{v}_2) = 5c_2\vec{v}_1 + (3c_1 - 4c_2)\vec{v}_2$

$\Rightarrow T(\vec{v}_1) = 3\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}_B$ and $T(\vec{v}_2) = 5\vec{v}_1 - 4\vec{v}_2 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}_B$

How are linear transformations impacted by a change of basis?

ex3: consider the shear transformation where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\vec{v}_1 + (c_1+c_2)\vec{v}_2$

What happens to $\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ under T?

$$\begin{tikzcd}
\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} -2 \\ 3 \end{bmatrix} \arrow[r, "T"] \arrow[d] & T(\vec{x}) = \begin{bmatrix} -3 \\ 17 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 3 \end{bmatrix} \arrow[d] \\
{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}_{\mathcal{B}}} \arrow[r, "B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}"] & {[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}}}
\end{tikzcd}$$

To find the B matrix

$B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

so $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Show the picture of overlaid grids.

$$\begin{tikzcd}
\vec{x} \arrow[r, "T \text{ or } A"] \arrow[d, shift left, "S^{-1}"] & T(\vec{x}) \arrow[d, shift left, "S^{-1}"] \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "B"] \arrow[u, shift left, "S"] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, shift left, "S"]
\end{tikzcd}$$

Notice we can get from $[\vec{x}]_{\mathcal{B}}$ to $T(\vec{x})$ via two routes.

$$AS[\vec{x}]_{\mathcal{B}} = SB[\vec{x}]_{\mathcal{B}} \Rightarrow A = SBS^{-1}$$

Dfn: Consider two $n \times n$ matrices A & B. We say that A is similar to B if there exists an invertible matrix S. s.t.

$$AS = SB \quad \text{OR} \quad A = SBS^{-1}.$$

Matrices are similar if they represent the same linear transformation WRT different bases.

ex4: Show $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ are not similar.

Let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Solve $AS \overset{?}{=} SB$.

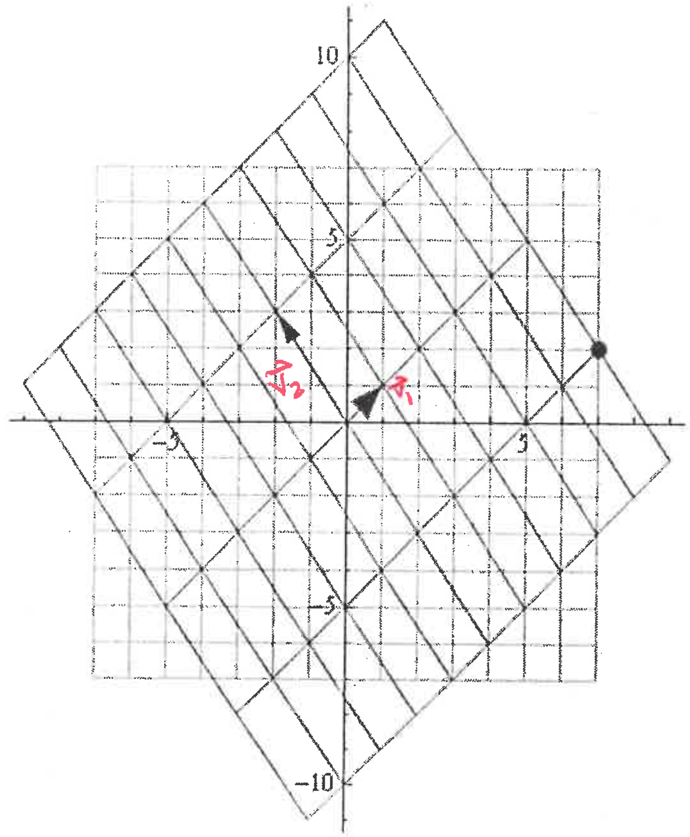
Thm: Similarity is an equivalence relation.

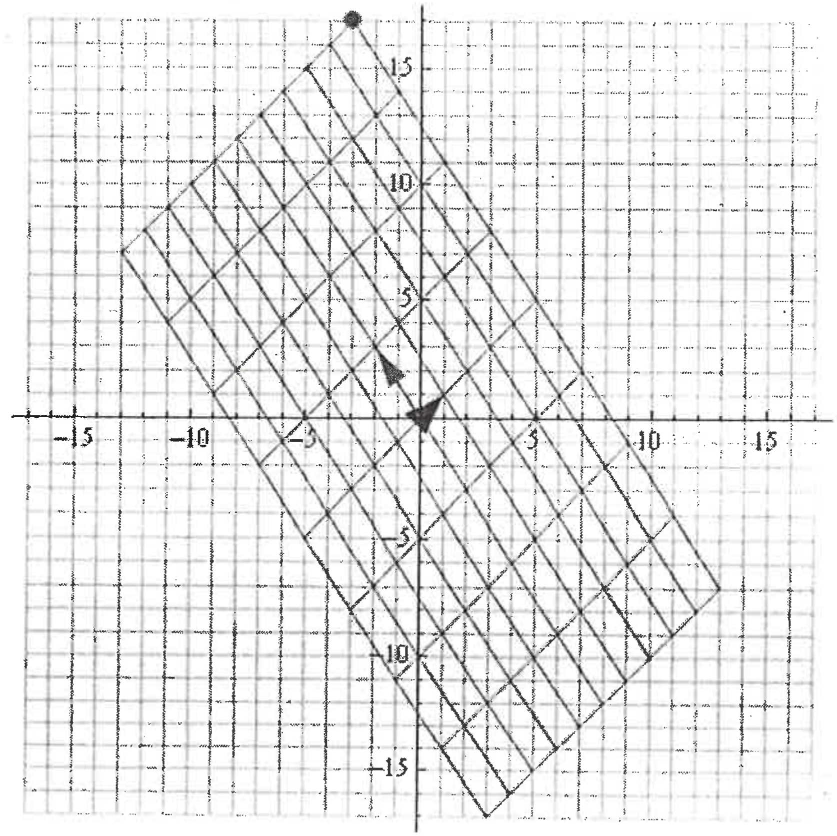
(a) An $n \times n$ matrix A is similar to itself (reflexivity)

(b) If A is similar to B, then B is similar to A (symmetry)

(c) If A is similar to B and B is similar to C, then A is similar to C (transitivity).

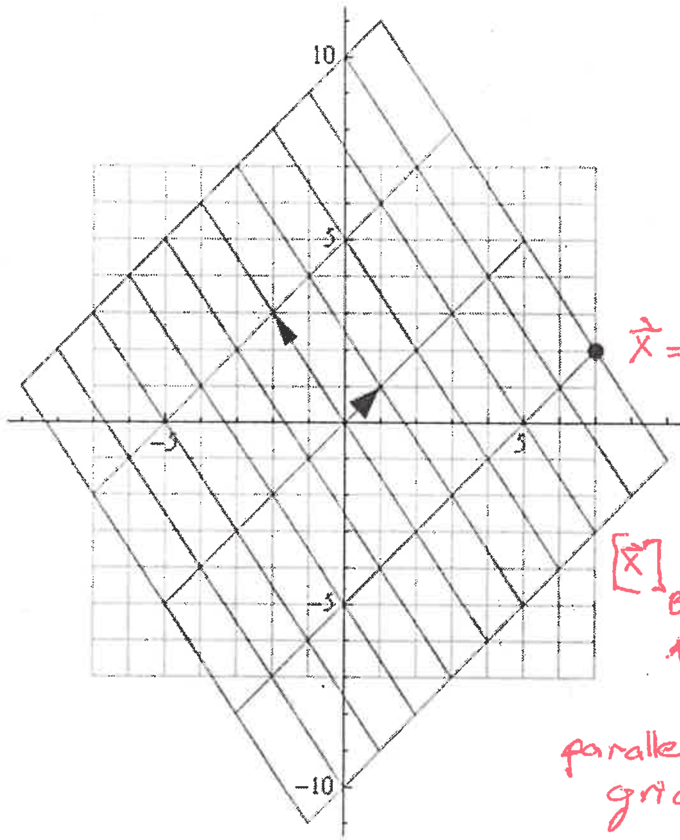
Linear Transformation using a change of basis





3.4
5/5

Linear Transformation using a change of basis



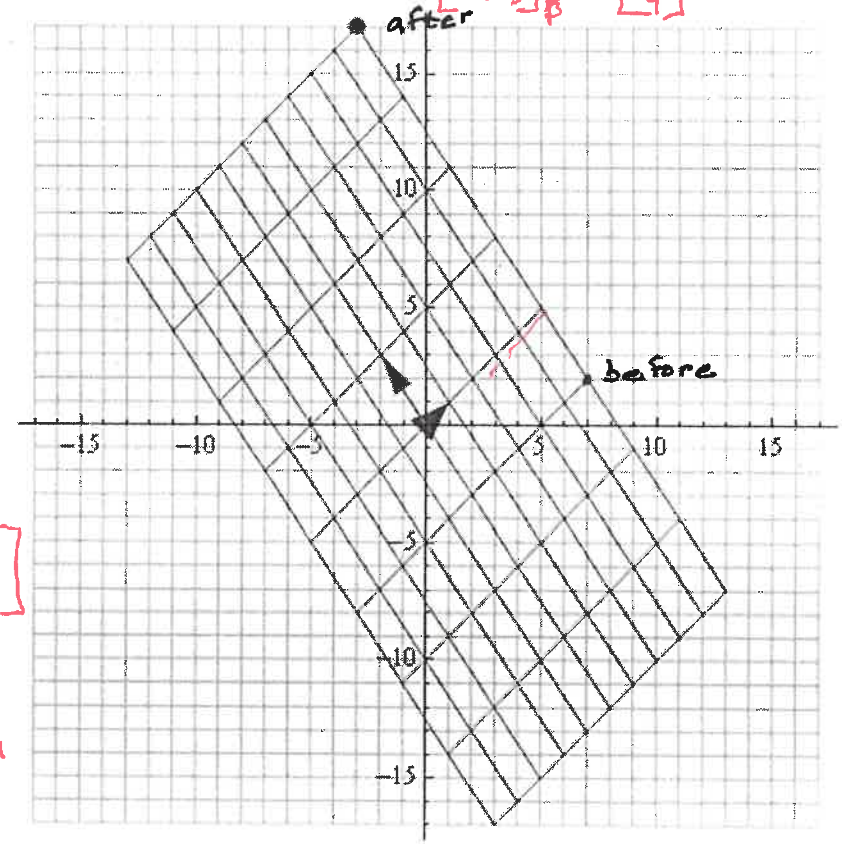
$\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$[\vec{x}]_B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

↑ parallelogram grid.

$T(\vec{x}) = \begin{bmatrix} -3 \\ 17 \end{bmatrix}$

$[T(\vec{x})]_B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$



3.4
5/5