3.3: Dimension of a Subspace of $\mathbb{R}^n$.

Intuitively, 2 vectors in $\mathbb{R}^3$ determine a plane (non-zero, not parallel). Thus we might guess (accurately) that the dimension of the plane is 2 ...

Recall: $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ are a basis for a subspace V provided they span and are L.I.

SPAN	L.I.
$\uparrow$	$\uparrow$
enough vectors.	no extra vectors.

__Thm:__ Consider vectors $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{\omega}_1, \dots, \vec{\omega}_q$ in a subspace V of $\mathbb{R}^n$. If $\vec{v}_1, \dots, \vec{v}_p$ are L.I. and $\vec{\omega}_1, \dots, \vec{\omega}_q$ span V, then $q \geq p$.

__Thm:__ All bases of a subspace V of $\mathbb{R}^n$ have the same number of vectors.

□ proof.
Consider 2 bases for V: $\vec{v}_1, \dots, \vec{v}_p$ & $\vec{\omega}_1, \dots, \vec{\omega}_q$.
$\vec{v}_1, \dots, \vec{v}_p$ are LI & $\vec{\omega}_1, \dots, \vec{\omega}_q$ span V so $p \leq q$

& $\vec{\omega}_1, \dots, \vec{\omega}_q$ are LI & $v_1, \dots, v_p$ span V so $q \leq p$

$\Rightarrow p = q$ QED ■

Think back to the basis for our plane..

Dfn. Consider a subspace V of $\mathbb{R}^n$. The number of vectors in a basis of V is called the <u>dimension</u> of V and denoted by $\dim(V)$.

Thm: Consider a subspace V of $\mathbb{R}^n$ w/ $\dim(V) = m$.

(a) We can find <u>at most</u> m L.I. vectors in V.

(b) We need <u>at least</u> m vectors to span V.

(c) If m vecs in V are LI, then they form a basis for V.

(d) If m vecs span V, then they form a basis for V.

□ proof of (d).

Assume $\text{span}(\vec{v}_1, \dots, \vec{v}_m) = V$. NTS $\vec{v}_1, \dots, \vec{v}_m$ are LI

Suppose $\vec{v}_1, \dots, \vec{v}_m$ are L.D.

$\Rightarrow \vec{v}_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1}$ ($\vec{v}_i$ is redundant).

$\Rightarrow \vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$ spans V

$\Rightarrow (m-1)$ vectors span V.

$\Rightarrow$ (b) is false. $\Rightarrow\Leftarrow$

$\therefore \vec{v}_1, \dots, \vec{v}_m$ are L.I.

ex1: Find a basis for the kernel & image of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}_{5\times 4}$$

(a) To find the kernel, solve $A\vec{x} = \vec{0}$

$$\text{rref}([A|\vec{0}]) = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ since x_3 is free, we replace it w/ the variable s.

All vectors in $\ker(A)$ are scalar multiples of $\vec{\omega}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

so $\vec{\omega}_1$ forms a basis for $\ker(A)$.

(b) To find the image, we must find all lin. comb. of the columns of A. How many columns are ~~redundant~~ i.e., correspond to free variables?

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{one row of zeros} \Rightarrow \text{one redundant column.}$$

better yet, the non-zero cols of $[\operatorname{rref}(A^T)]^T$ form a basis for $\operatorname{Im}(A)$.

$$\text{so} \quad \vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

are a basis...

An alternate method... $\operatorname{rref}(A)$ and Identify the cols. w/ a leading 1. These cols in A form a basis for $\operatorname{Im}(A)$.

$$\text{so} \quad \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}; \quad \vec{b}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}; \quad \vec{b}_4 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

are a basis for $\operatorname{Im}(A)$.

Think about it this way, each col w/ a leading 1 in rref(A) corresponds to basis vec. in im(A). The other cols give you free variables and basis vecs for ker(A).

If $A_{n \times m}$, $m = \underbrace{\dim(\mathrm{im}(A))}_{\mathrm{rank}(A)} + \underbrace{\dim(\ker(A))}_{\mathrm{nullity}(A)}$.

So $T(\vec{x}) = A\vec{x}$ where $T: \mathbb{R}^m \mapsto \mathbb{R}^n$ and $A_{n \times m}$ sends Nullity(A) cols to the Kernel(A)/$\vec{0}$ & the lin. comb. of the remaining form the image.

ex2: Find a basis for im(A) & ker(A) of

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix} \Rightarrow \mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{im}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right)$$

$$\ker(A) = \mathrm{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

Thm: $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ are a basis for $\mathbb{R}^n$ iff $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ is invertible.