3.2: Subspaces, bases, & L.I.

Dfn A subset W of the vector space $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if it has the following properties.

(a) W contains the zero vec. in $\mathbb{R}^n$.

(b) W is closed under addition.

$$(\vec{w}_1, \vec{w}_2 \in W \Rightarrow \vec{w}_1 + \vec{w}_2 \in W)$$

(c) W is closed under scalar mult.

$$(\vec{w}_1 \in W \text{ \& } k \in \mathbb{R} \Rightarrow k\vec{w}_1 \in W)$$

Note: If $\underset{n \times m}{A\vec{x}}$ is a lin. trans from $\mathbb{R}^m \mapsto \mathbb{R}^n$; then $\underbrace{\ker(A)}_{\subset \mathbb{R}^m}$ & $\underbrace{\operatorname{im}(A)}_{\subset \mathbb{R}^n}$ are subspaces

Ex 1: Show $W = \left\{ \vec{x} \mid \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix},\ x_1, x_2 \in \mathbb{R} \right\}$ is not a subspace

Students should review (ex.2) carefully in the book.

Ex 2: In 3.1 we saw that the im(A) where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ is } \operatorname{im}(A) = \operatorname{span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} \right\}$$

The span is a plane and so $\operatorname{im}(A) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$
$= \operatorname{span}(\vec{v}_1, \vec{v}_2)$

Note that $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$. That is the redundant vector is a lin. comb. of $\vec{v}_1, \vec{v}_2$.

Dfn: Consider $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$

(a) We say $\vec{v}_i$ in $\vec{v}_1, \dots, \vec{v}_m$ is redundant if $\vec{v}_i$ is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_{i-1}$.

NOTE: We call $\vec{v}_1$ redundant if $\vec{v}_1 = \vec{0}$

(b) $\vec{v}_1, \dots, \vec{v}_m$ are L.I if none are redundant.

(c) $\vec{v}_1, \dots, \vec{v}_m$ form a basis for V if they span V & are L.I.

ex2 rev: $\vec{v}_1$ & $\vec{v}_2$ are L.I. and a basis for $im(A)$

How do you find out whether vectors are L.I.? Sure we can do it by inspection... but perhaps we can do better. If they are L.I., then no vector is a linear combination of the others ...

$$\text{rref}\left(\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\right)$$ has at least 1 column w/o a pivot.

Dfn: Consider $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$. An eqt. of the form

$$c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$$ is called a relation among $\vec{v}_1, \dots, \vec{v}_m$.

Trivial vs. nontrivial solutions.

Thm: $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ are L.D. iff there are nontrivial relations among them.

□ proof.

($\Rightarrow$) Assume $\vec{v}_1, \dots, \vec{v}_m$ are L.D.

$\Rightarrow$ $\vec{v}_i$ is redundant. That is $V_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1}$

$\Rightarrow$ $c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i = \vec{0}$ (a non trivial sol.).

($\Leftarrow$) Assume there are nontrivial relations among $\vec{v}_1, \dots, \vec{v}_m$.

$\Rightarrow$ There is a nontrivial sol. to $c_1\vec{v}_1 + \dots + c_i\vec{v}_i + \dots + c_m\vec{v}_m = \vec{0}$ where i is the highest index s.t. $c_i \neq 0$.

$\Rightarrow$ $\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - \dots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1}$

$\Rightarrow$ $\vec{v}_i$ is a lin. comb of the other vecs and redundant and so $\vec{v}_1, \dots, \vec{v}_m$ is a L.D. set. ■

ex3: If the cols. of $A_{n\times m}$ are L.I., find ker(A).

Solve $A\vec{x} = \vec{0}$ $\Rightarrow$ $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \vec{x} = \vec{0}$

$\Rightarrow$ $x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{0}$

Since $\vec{v}_1, \dots, \vec{v}_m$ are L.I., there is only the the trivial sol. and $\ker(A) = \{\vec{0}\}$.

Thus we see that the cols of A are L.I. iff $\ker(A) = \{\vec{0}\}$

Equivalent statements about L.I.: ($\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$)

(i) $\vec{v}_1, \dots, \vec{v}_m$ are L.I.

(ii) None of $\vec{v}_1, \dots, \vec{v}_m$ are redundant.
or a lin. comb. of the others.

(iii) $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ has only the triv. sol.

(iv) $\ker\left(\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\right) = \{\vec{0}\}$

(v) $\operatorname{rank}\left(\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\right) = m.$

Thm: Consider $\vec{v}_1, \dots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$.
$\vec{v}_1, \dots, \vec{v}_m$ are a basis for V iff all vectors $\vec{v} \in V$ can be expressed as a unique lin. comb. of $\vec{v}_1, \dots, \vec{v}_m$.

□ proof

($\Rightarrow$) Assume $\vec{v}_1, \dots, \vec{v}_m$ is a basis for V and that there is more than 1 lin comb. that represents $\vec{v} \in V$.

$$\Rightarrow c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{v}$$
and
$$d_1\vec{v}_1 + \dots + d_m\vec{v}_m = \vec{v}$$

$$\Rightarrow (c_1 - d_1)\vec{v}_1 + \dots + (c_m - d_m)\vec{v}_m = \vec{0}$$

$\Rightarrow c_1 - d_1 = \dots = c_m - d_m = 0$ since $\vec{v}_1, \dots, \vec{v}_m$ is a basis/L.I.

$\Rightarrow c_1 = d_1 \quad \dots \quad c_m = d_m$ & the representation is unique.

($\Leftarrow$) Assume all $\vec{v} \in V$ are represented uniquely by a lin. comb. of $\vec{v}_1, \dots, \vec{v}_m$.

$\Rightarrow c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ has a unique sol.
... the triv. sol.

$\Rightarrow \vec{v}_1, \dots, \vec{v}_m$ are L.I & form a basis for V. ■