2.4: Inverse of a Matrix

recall: $ax = b$

$\Leftrightarrow \; x = a^{-1}b, \; a \neq 0$

where a^{-1} is the multiplicative inverse of a

What if $A\vec{x} = \vec{b}$?

Dfn: An $n \times n$ matrix A is invertible (nonsingular) if $\exists \; B_{n \times n}$ s.t. $AB = BA = I_n$.

B is called the (multiplicative) inverse of A.

If no such B exists, we say A is singular.

NOTE: A must be square to be invertible.

Thm: The inverse is unique.

$\square$ proof (by contradiction)

Assume the inverse to A is not unique.

$\Rightarrow \exists \; B, C$, s.t. $AB = I_n = BA$ & $AC = I_n = CA$ and $B \neq C$.

Now $AB = I$

$\Rightarrow C(AB) = CI$

$\Rightarrow (CA)B = C$

$\Rightarrow IB = C$

$\Rightarrow B = C \quad \Rightarrow\Leftarrow$

therefore the inverse is unique. we call it A^{-1}.

Notice that $S(\vec{x}) = A^{-1}\vec{x}$ is a linear transformation.

If $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = A^{-1}\vec{x}$

$\Rightarrow T(S(\vec{x})) = T(A^{-1}\vec{x}) = A(A^{-1}\vec{x}) = \vec{x}$

and $S(T(\vec{x})) = S(A\vec{x}) = A^{-1}(A\vec{x}) = \vec{x}$

So S is the inverse linear transformation of T aka T^{-1}.

ex1: show that A^{-1} exists

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

(see ex3 b&c on p 26).

thm: $A^{-1}_{n\times n}$ exists iff

- $\text{rref}(A) = I_n$
- $\text{rank}(A) = n$

Thm:
- $A_{n\times n}\vec{x} = \vec{b}$ has a unique sol. if A^{-1} exists, otherwise it has infinitely many or none.
- $A\vec{x} = \vec{0}$ has only the trivial sol. $\vec{x} = \vec{0}$ when A^{-1} exists. If A is not invertible, then there are infinitely many sols.

These follow from Thm 1,3,4 and example 1,3,3d.

How to find the inverse of a matrix.

key concept: $A: \vec{x} \longmapsto \vec{b}$ and $A^{-1}: \vec{b} \longmapsto \vec{x}$

Ex 1 new: Solve $A\vec{x} = \vec{b}$ for an arbitrary $\vec{b}$.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \Rightarrow \left| \begin{array}{rrrcl} x_1 & -x_2 & & = & b_1 \\ x_1 & & -x_3 & = & b_2 \\ -6x_1 & +2x^2 & +3x_3 & = & b_3 \end{array} \right|$$

So $$\operatorname{rref}\left(\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \right)$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

meaning
$x_1 = -2b_1 - 3b_2 - b_3$
$x_2 = -3b_1 - 3b_2 - b_3$
$x_3 = -2b_1 - 4b_2 - b_3$

method: $\operatorname{rref}([A \mid I]) = [I \mid A^{-1}]$.
provided A^{-1} exists.
If you don't get I then A is singular.

What about the product of invertible matrices?

suppose $A_{n\times n}$ & $B_{n\times n}$ are invertible
and that $(AB)\vec{x} = \vec{y}$

$$\Rightarrow A^{-1}AB\vec{x} = B\vec{x} = A^{-1}\vec{y}$$
$$\Rightarrow B^{-1}B\vec{x} = \vec{x} = (B^{-1}A^{-1})\vec{y}$$

Hence $(AB)^{-1} = B^{-1}A^{-1}$ (order matters).

<u>Thm</u>: Let A, B be $n \times n$ matrices s.t.

$BA = I_n$. Then

(a) A & B are both invertible

(b) $A^{-1} = B$ & $B^{-1} = A$.

(c) $AB = I_n$.

□ <u>proof</u>.

It suffices to show $AB = I_n$ (see 1st thm in section).

To see this, consider $A\vec{x} = \vec{0}$

$$\Rightarrow BA\vec{x} = B\vec{0}$$

$$\Rightarrow I\vec{x} = \vec{0}$$

$$\Rightarrow \vec{x} = \vec{0}$$

Since $\vec{x} = \vec{0}$ is the only soln, A^{-1} exists.

now $AB = I \Rightarrow A^{-1}AB = A^{-1}I \Rightarrow B = A^{-1}$!

and $AB = AA^{-1} = I$. ∎

ex 2: Find the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we call $ad-bc$ the determinant of $A_{2\times2}$.

so $A^{-1}_{2\times2}$ only exists when $\det(A) \neq 0$.

What does $\det(A_{2\times2})$ represent? It gives the area of the parallelogram determined by the cols. of A.

To see this, recall $|\vec{v} \times \vec{w}|$ = area of the parallelogram determined by $\vec{v}, \vec{w}$

where $\vec{v}, \vec{w} \in \mathbb{R}^3$.

Find $\begin{bmatrix} a \\ c \\ 0 \end{bmatrix} \times \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & c & 0 \\ b & d & 0 \end{vmatrix}$

$$= 0\vec{i} + 0\vec{j} + (ad-bc)\vec{k}$$

w/ magnitude $(ad-bc)$