1.3: On the solution of Linear Systems and matrix algebra

I Prelim

- After Gauss-Jordan Elimination we say a matrix is in reduced row echelon form or RREF.

- Given matrix A, we can find rref(A)

- Dfn: Rank

  we define the rank of matrix A, or rank(A), as the number of leading 1's (pivots) in rref(A).

II Solutions

- inconsistent
  - no soln.
  - [0 . . . . 0 | 1]
- consistent
  - infinitely many solutions
    - free variable(s)
      - at least 1 col. of rref(A) w/o a leading 1.
  - one (unique) soln.
    - all cols of rref(A) have a pivot

- read carefully thm ex 1-5 and Thm 1.3.3 & 1.3.4

II matrix algebra

- adding matrices (by element)

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & & a_{nm} \end{bmatrix}_{n\times m} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}_{n\times m} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}_{n\times m}$$

- multiplying a matrix by a scalar. (by element/entry)

$$k\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}$$

- Dot product: If $\vec{u}, \vec{v}$ are row or col. vectors w/ components $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ then
  $\vec{u}\cdot\vec{v} = u_1v_1 + \ldots + u_nv_n$
  ex: $\vec{w}\cdot\vec{v} = \begin{bmatrix} 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

- Two ways to write $A\vec{x}$
  If $A_{n\times m}$ w/ rows $\vec{w}_1, \ldots, \vec{w}_n$ and $\vec{x} \in \mathbb{R}^m$ then

① $A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ - & \vec{w}_2 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1\cdot\vec{x} \\ \vec{w}_2\cdot\vec{x} \\ \vdots \\ \vec{w}_n\cdot\vec{x} \end{bmatrix}$

ex: $A\vec{x} = \begin{bmatrix} 2 & 6 \\ 0 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 2(1) + 6(4) \\ 0(1) + 4(4) \\ 1(1) + 2(4) \end{bmatrix}$$

$$= 1\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$\uparrow$ $\uparrow$

columns of A: $\vec{v}_1$ and $\vec{v}_2$

② and $A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$

$$= x_1\vec{v}_1 + \ldots + x\vec{v}_m$$

Dfn: A vector $\vec{b} \in \mathbb{R}^n$ is called a linear combination of $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ if there exist scalars $x_1, \ldots, x_m$ s.t. $\vec{b} = x_1\vec{v}_1 + \ldots + x_m\vec{v}_m$.

so $A\vec{x}$ is a linear combo of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

This will be used extensively in the theoretical areas. We can see ...

$A\vec{x} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$ (you can break apart $A\vec{x}$)

and

$x_1\vec{v}_1 + \dots + x_n\vec{v}_m = A\vec{x}$ (you can combine a lin. combo into $A\vec{x}$).

<u>Two rules for $A\vec{x}$</u>: If A is an $n \times m$ matrix and $\vec{x}, \vec{y} \in \mathbb{R}^n$, and k is a scalar

(a) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

(b) $A(k\vec{x}) = kA\vec{x}$

□ proof of (a) ... alternate prf is clearer.

$$A(k\vec{x}) = A\left(k\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}\right)$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} kx_1 \\ \vdots \\ kx_m \end{bmatrix}$$

$$= kx_1\vec{v}_1 + \dots + kx_m\vec{v}_m$$

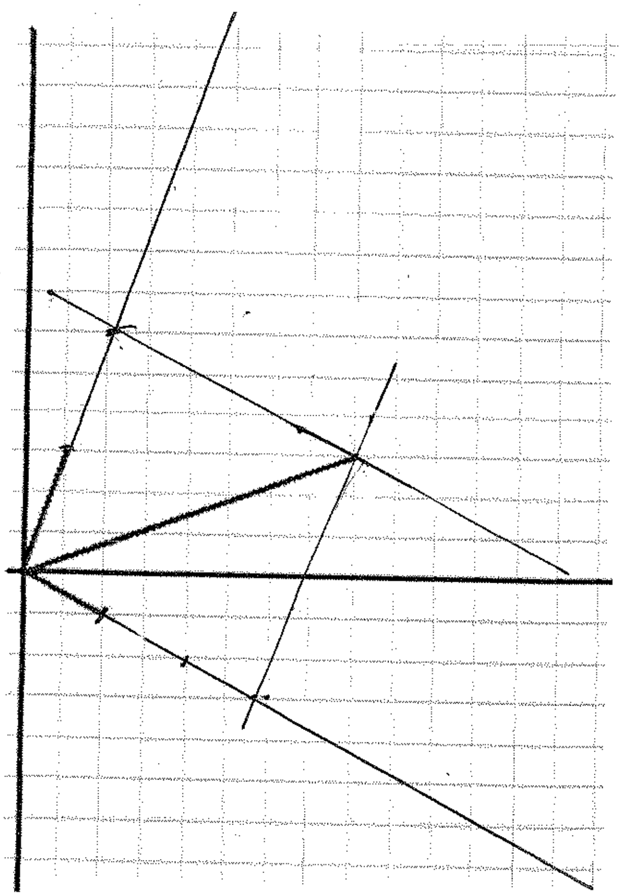
$$= k(x_1\vec{v}_1 + \dots + x_m\vec{v}_m)$$

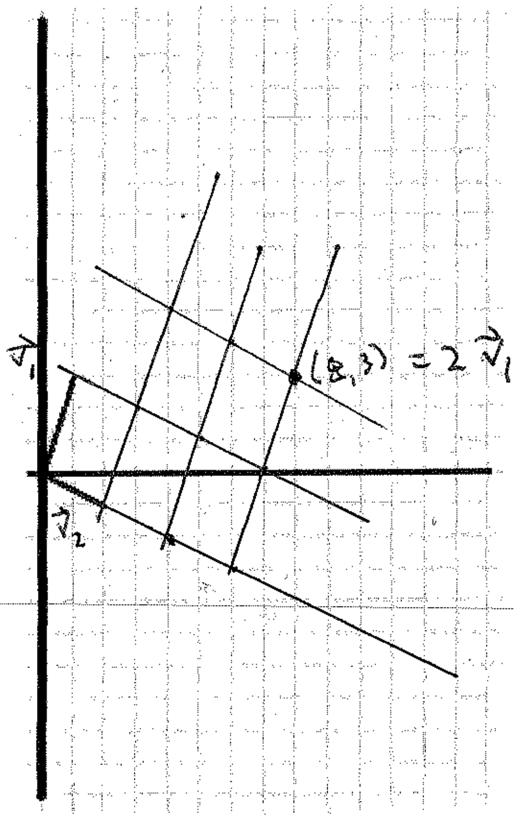
$$= k\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = kA\vec{x} \quad \blacksquare$$

Linear System: $\left| \begin{array}{l} x_1 + 2x_2 = 8 \\ 3x_1 - x_2 = 3 \end{array} \right|$

w/ augmented matrix $[A \mid b]$ $\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -1 & 3 \end{array}\right]$

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$




$\vec{v}_1$
$\vec{v}_2$
$(8,3) = 2\vec{v}_1 + 3\vec{v}_2$