3.4.82 a. We seek the real numbers x_2, x_3, and c such that $T\begin{bmatrix}1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_2\\x_3\\3x_3-2x_2\end{bmatrix}=c\begin{bmatrix}1\\x_2\\x_3\end{bmatrix}$. Examining the components of this vector equation, we find $x_2=c$, $x_3=x_2^2$ and $3x_2^2-2x_2=x_2^3$. Writing the last equation as $x_2\left(x_2^2-3x_2+2\right)=x_2(x_2-1)(x_2-2)=0$, we find the three solutions $x_2=0,\ 1,2$. The vectors we seek are $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\2\\4\end{bmatrix}$. b. By Theorem 3.4.7, the three vectors we found in part a will do the job. (Check that these vectors are linearly independent.)

True or False

Ch 3.TF.1 T, by Theorem 3.3.2.

Ch 3.TF.2 F; The nullity is $6-4=2$, by Theorem 3.3.7.

Ch 3.TF.3 F; It's a subspace of $\mathbb{R}^3$.

Ch 3.TF.4 T; by Definition 3.1.2.

Ch 3.TF.5 T, by Summary 3.3.10.

Ch 3.TF.6 F, by Theorem 3.3.7.

Ch 3.TF.7 T, by Summary 3.3.10.

Ch 3.TF.8 F; The identity matrix is similar only to itself.

Ch 3.TF.9 T; We have the nontrivial relation $3\vec{u}+3\vec{v}+3\vec{w}=\vec{0}$.

Ch 3.TF.10 F; The columns could be $\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4$ in $\mathbb{R}^5$, for example.

Ch 3.TF.11 T, by Theorem 3.4.6, parts b and c.

Ch 3.TF.12 F; Let $V=\text{span}\begin{bmatrix}1\\1\end{bmatrix}$ in $\mathbb{R}^2$, for example.

Ch 3.TF.13 T, by Definition 3.2.3.

Ch 3.TF.14 T, by Definition 3.2.1.

Ch 3.TF.15 T; Check that $\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&1\\-1&1\end{bmatrix}=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

Ch 3.TF.16 T, by Theorem 3.3.9.

Ch 3.TF.17 T, by Theorem 3.2.8.

Ch 3.TF.18 T, by Summary 3.3.10.

Ch 3.TF.19 F; The number n may exceed 4.

Ch 3.TF.20 T, by Definition 3.2.1 (V is closed under linear combinations)

Ch 3.TF.21 T, since $A^{-1}(AB)A = BA$.

Ch 3.TF.22 T, since both kernels consist of the zero vector alone.

Ch 3.TF.23 F; There is no invertible matrix S as required in the definition of similarity.

Ch 3.TF.24 F; Five vectors in $\mathbb{R}^4$ must be dependent, by Theorem 3.2.8.

Ch 3.TF.25 T, by Definition 3.2.1 (all vectors in $\mathbb{R}^3$ are linear combinations of $\vec{e}_1, \vec{e}_2, \vec{e}_3$).

Ch 3.TF.26 T; Use a basis with one vector on the line and the other perpendicular to it.

Ch 3.TF.27 T, since $AB\vec{v} = A\vec{0} = \vec{0}$.

Ch 3.TF.28 T, by Definition 3.2.3.

Ch 3.TF.29 F; Suppose $\vec{v}_2 = 2\vec{v}_1$. Then $T(\vec{v}_2) = 2T(\vec{v}_1) = 2\vec{e}_1$ cannot be $\vec{e}_2$.

Ch 3.TF.30 F; Consider $\vec{u} = \vec{e}_1,\ \vec{v} = 2\vec{e}_1$, and $\vec{w} = \vec{e}_2$.

Ch 3.TF.31 F; Note that $\mathbb{R}^2$ isn't even a subset of $\mathbb{R}^3$. A vector in $\mathbb{R}^2$, with two components, does not belong to $\mathbb{R}^3$.

Ch 3.TF.32 T; If $B = S^{-1}AS$, then $B + 7I_n = S^{-1}(A + 7I_n)S$.

Ch 3.TF.33 T; for any $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of V also $k\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis too, for any nonzero scalar k.

Ch 3.TF.34 F; The identity matrix is similar only to itself.

Ch 3.TF.35 F; Consider $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

Ch 3.TF.36 F; Let $A = I_2,\ B = -I_2$ and $\vec{v} = \vec{e}_1$, for example.

Ch 3.TF.37 F; Let $V = \text{span}(\vec{e}_1)$ and $W = \text{span}(\vec{e}_2)$ in $\mathbb{R}^2$, for example.

Ch 3.TF.38 T; If $A\vec{v} = A\vec{w}$, then $A(\vec{v} - \vec{w}) = \vec{0}$, so that $\vec{v} - \vec{w} = \vec{0}$ and $\vec{v} = \vec{w}$.

Ch 3.TF.39 T; Consider the linear transformation with matrix $[\vec{w}_1 \ \dots \ \vec{w}_n][\vec{v}_1 \ \dots \ \vec{v}_n]^{-1}$.

Ch 3.TF.40 F; Suppose A were similar to B. Then $A^4 = I_2$ were similar to $B^4 = -I_2$, by Example 7 of Section 3.4. But this isn't the case: I_2 is similar only to itself.

Ch 3.TF.41 T; Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example, with $\ker(A) = \text{im}(A) = \text{span}(\vec{e}_1)$.

Ch 3.TF.42 F; Consider I_n and $2I_n$, for example.

Ch 3.TF.43 T; Matrix $B = S^{-1}AS$ is invertible, being the product of invertible matrices.

Ch 3.TF.44 T; Note that $\text{im}(A)$ is a subspace of $\ker(A)$, so that

$\dim(\text{im } A) = \text{rank}(A) \leq \dim(\ker A) = 10 - \text{rank}(A)$.

Ch 3.TF.45 T; Pick three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ that span V. Then $V = \text{im}[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$.

Ch 3.TF.46 T; Check that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

Ch 3.TF.47 T; Pick a vector $\vec{v}$ that is neither on the line nor perpendicular to it. Then the matrix of the linear transformation $T(\vec{x}) = R\vec{x}$ with respect to the basis $\vec{v}$, $R\vec{v}$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since $R(R\vec{v}) = \vec{v}$.

Ch 3.TF.48 F; If $B = S^{-1}AS$, then $B = (2S)^{-1}A(2S)$ as well.

Ch 3.TF.49 T; Note that $A(B - C) = 0$, so that all the columns of matrix $B - C$ are in the kernel of A. Thus $B - C = 0$ and $B = C$, as claimed.

Ch 3.TF.50 T; Suppose $\vec{v}$ is in both $\ker(A)$ and $\text{im}(A)$, so that $\vec{v} = A\vec{w}$ for some vector $\vec{w}$. Then $\vec{0} = A\vec{v} = A^2\vec{w} = A\vec{w} = \vec{v}$, as claimed.

Ch 3.TF.51 F; Suppose such a matrix A exists. Then there is a vector $\vec{v}$ in $\mathbb{R}^2$ such that $A^2\vec{v} \neq \vec{0}$ but $A^3\vec{v} = \vec{0}$. As in Exercise 3.4.58a we can show that vectors $\vec{v}, A\vec{v}, A^2\vec{v}$ are linearly independent, a contradiction (we are looking at three vectors in $\mathbb{R}^2$).

Ch 3.TF.52 T; The ith column $\vec{a}_i$ of A, being in the image of A, is also in the image of B, so that $\vec{a}_i = B\vec{c}_i$ for some $\vec{c}_i$ in $\mathbb{R}^m$. If we let $C = [\vec{c}_1\ \cdots\ \vec{c}_m]$, then $BC = [B\vec{c}_1\ \cdots\ B\vec{c}_m] = [\vec{a}_1\ \cdots\ \vec{a}_m] = A$, as required.

Ch 3.TF.53 F; Think about this problem in terms of "building" such an invertible matrix column by column. If we wish the matrix to be invertible, then the first column can be any column other than $\vec{0}$ (7 choices). Then the second column can be any column other than $\vec{0}$ or the first column (6 choices). For the third column, we have *at most* 5 choices (not $\vec{0}$ or the first or second columns, as well as possibly some other columns). For some choices of the first two columns there will be other columns we have to exclude (the sum or difference of the first two), but not for others. Thus, in total, fewer than $7 \times 6 \times 5 = 210$ matrices are invertible, out of a total $2^9 = 512$ matrices. Thus, most are not invertible.