of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for any positive integer m. See Exercise 81.

d. In your equation in part (c), let m go to infinity to find $\lim_{m\to\infty} (A^m \vec{x})$. Verify that your answer is the equilibrium distribution for A.

83. If $A\vec{x} = \vec{x}$ for a regular transition matrix A and a distribution vector $\vec{x}$, show that all components of $\vec{x}$ must be positive. (Here you are proving the last claim rem 2.3.11a.)

84. Consider an $n \times m$ matrix A of rank n. Show exists an $m \times n$ matrix X such that $AX = I_n$. how many such matrices X are there?

85. Consider an $n \times n$ matrix A of rank n. How m matrices X are there such that $AX = I_n$?

2.4 The Inverse of a Linear Transformation

Let's first review the concept of an invertible function. As you read thes definitions, consider the examples in Figures 1 and 2, where X and Y are f

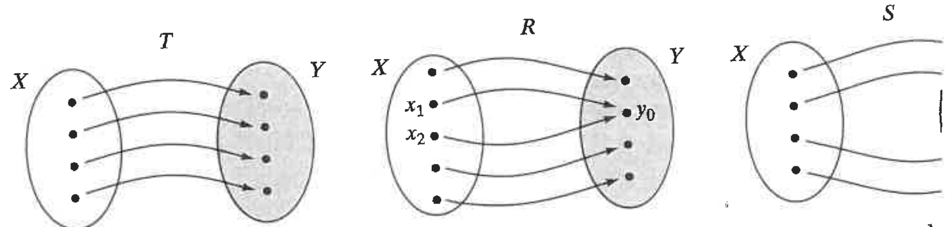


Figure 1 T is invertible. R is not invertible: The equation $R(x) = y_0$ has two solutions, x_1 and invertible: There is no x such that $S(x) = y_0$.

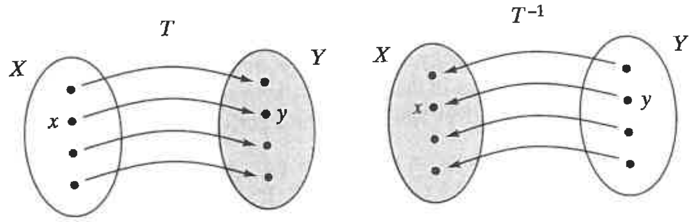


Figure 2 A function T and its inverse T^{-1}.

Definition 2.4.1

Invertible Functions

A function T from X to Y is called invertible if the equation $T(x)$ unique solution x in X for each y in Y.

In this case, the inverse T^{-1} from Y to X is defined by

$$T^{-1}(y) = \big(\text{the unique } x \text{ in } X \text{ such that } T(x) = y\big).$$

To put it differently, the equation

$$x = T^{-1}(y) \quad \text{means that} \quad y = T(x).$$

Note that

$$T^{-1}\big(T(x)\big) = x \quad \text{and} \quad T\big(T^{-1}(y)\big) = y$$

for all x in X and for all y in Y.

Conversely, if L is a function from Y to X such that

$$L\big(T(x)\big) = x \quad \text{and} \quad T\big(L(y)\big) = y$$

for all x in X and for all y in Y, then T is invertible and $T^{-1} = L$.

If a function T is invertible, then so is T^{-1} and $(T^{-1})^{-1} = T$.

If a function is given by a formula, we may be able to find the inverse by solving the formula for the input variable(s). For example, the inverse of the function

$$y = \frac{x^3 - 1}{5} \quad \text{(from } \mathbb{R} \text{ to } \mathbb{R}\text{)}$$

is

$$x = \sqrt[3]{5y + 1}.$$

Now consider the case of a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by

$$\vec{y} = T(\vec{x}) = A\vec{x},$$

where A is an $n \times n$ matrix. (The case of an $n \times m$ matrix will be discussed in Exercise 48.)

According to Definition 2.4.1, the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible if the *linear system*

$$A\vec{x} = \vec{y}$$

has a unique solution $\vec{x}$ in $\mathbb{R}^n$ for all $\vec{y}$ in the vector space $\mathbb{R}^n$. By Theorem 1.3.4, this is the case if (and only if) $\text{rank}(A) = n$ or, equivalently, if

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = I_n.$$

Definition 2.4.2 **Invertible matrices**

A square matrix A is said to be *invertible* if the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible. In this case, the matrix[10] of T^{-1} is denoted by A^{-1}. If the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible, then its inverse is $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$.

Theorem 2.4.3 **Invertibility**

An $n \times n$ matrix A is invertible if (and only if)

$$\text{rref}(A) = I_n$$

or, equivalently, if

$$\text{rank}(A) = n.$$

The following proposition follows directly from Theorem 1.3.4 and Example 1.3.3d.

Theorem 2.4.4 **Invertibility and linear systems**

Let A be an $n \times n$ matrix.

a. Consider a vector $\vec{b}$ in $\mathbb{R}^n$. If A is invertible, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$. If A is noninvertible, then the system $A\vec{x} = \vec{b}$ has infinitely many solutions or none.

[10]The inverse transformation is linear. See Exercise 2.2.29.

Theorem 2.4.4 **Invertibility and linear systems *(Continued)***

b. Consider the special case when $\vec{b} = \vec{0}$. The system $A\vec{x} = \vec{0}$ has ... as a solution. If A is invertible, then this is the only solution. ... noninvertible, then the system $A\vec{x} = \vec{0}$ has infinitely many soluti...

EXAMPLE 1

Is the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

invertible?

Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \begin{matrix} \\ -2(\mathrm{I}) \\ -3(\mathrm{I}) \end{matrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix} \begin{matrix} -(\mathrm{II}) \\ \\ -5(\mathrm{II}) \end{matrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ \div(-1) \end{matrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} -(\mathrm{III}) \\ \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 =$$

Matrix A is invertible since $\mathrm{rref}(A) = I_3$.

Let's find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

in Example 1 or, equivalently, the inverse of the linear transformation

$$\vec{y} = A\vec{x} \quad \text{or} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + 2x_3 \\ 3x_1 + 8x_2 + 2x_3 \end{bmatrix}.$$

To find the inverse transformation, we solve this system for the input ... x_2, and x_3:

$$\left| \begin{array}{rcl} x_1 + x_2 + x_3 & = & y_1 \\ 2x_1 + 3x_2 + 2x_3 & = & y_2 \\ 3x_1 + 8x_2 + 2x_3 & = & y_3 \end{array} \right| \begin{matrix} \longrightarrow \\ -2(\mathrm{I}) \\ -3(\mathrm{I}) \end{matrix}$$

$$\left| \begin{array}{rcl} x_1 + x_2 + x_3 & = & y_1 \\ x_2 & = & -2y_1 + y_2 \\ 5x_2 - x_3 & = & -3y_1 + y_3 \end{array} \right| \begin{matrix} -(\mathrm{II}) \\ \longrightarrow \\ -5(\mathrm{II}) \end{matrix}$$

$$\left| \begin{array}{rcl} x_1 + x_3 & = & 3y_1 - y_2 \\ x_2 & = & -2y_1 + y_2 \\ -x_3 & = & 7y_1 - 5y_2 + y_3 \end{array} \right| \begin{matrix} \longrightarrow \\ \\ \div(-1) \end{matrix}$$

$$\left| \begin{array}{rcl} x_1 + x_3 & = & 3y_1 - y_2 \\ x_2 & = & -2y_1 + y_2 \\ x_3 & = & -7y_1 + 5y_2 - y_3 \end{array} \right| \begin{matrix} -(\mathrm{III}) \\ \longrightarrow \\ \\ \end{matrix}$$

$$\left| \begin{array}{rcl} x_1 & = & 10y_1 - 6y_2 + y_3 \\ x_2 & = & -2y_1 + y_2 \\ x_3 & = & -7y_1 + 5y_2 - y_3 \end{array} \right|.$$

We have found the inverse transformation; its matrix is

$$B = A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}.$$

We can write the preceding computations in matrix form:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \longrightarrow \\ -2(\mathrm{I}) \\ -3(\mathrm{I}) \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \begin{array}{c} -(\mathrm{II}) \\ \longrightarrow \\ -5(\mathrm{II}) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right] \begin{array}{c} \longrightarrow \\ \\ \div(-1) \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \begin{array}{c} -(\mathrm{III}) \\ \longrightarrow \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right].$$

This process can be described succinctly as follows.

Theorem 2.4.5 **Finding the inverse of a matrix**

To find the *inverse* of an $n \times n$ matrix A, form the $n \times (2n)$ matrix $\left[A \mid I_n\right]$ and compute $\operatorname{rref}\left[A \mid I_n\right]$.

- If $\operatorname{rref}\left[A \mid I_n\right]$ is of the form $\left[I_n \mid B\right]$, then A is invertible, and $A^{-1} = B$.
- If $\operatorname{rref}\left[A \mid I_n\right]$ is of another form (i.e., its left half fails to be I_n), then A is not invertible. Note that the left half of $\operatorname{rref}\left[A \mid I_n\right]$ is $\operatorname{rref}(A)$.

Next let's discuss some algebraic rules for matrix inversion.

- Consider an invertible linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. By Definition 2.4.1, the equation $T^{-1}(T(\vec{x})) = \vec{x}$ holds for all $\vec{x}$ in $\mathbb{R}^n$. Written in matrix form, this equation reads $A^{-1}A\vec{x} = \vec{x} = I_n\vec{x}$. It follows that $A^{-1}A = I_n$. Likewise, we can show that $AA^{-1} = I_n$.

Theorem 2.4.6 **Multiplying with the inverse**

For an invertible $n \times n$ matrix A,

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n.$$

- If A and B are invertible $n \times n$ matrices, is BA invertible as well? If so, what is its inverse?

  To find the inverse of the linear transformation

  $$\vec{y} = BA\vec{x},$$

  we solve the equation for $\vec{x}$ in two steps. First, we multiply both sides of the equation by B^{-1} from the left:

  $$B^{-1}\vec{y} = B^{-1}BA\vec{x} = I_nA\vec{x} = A\vec{x}.$$

  Now, we multiply by A^{-1} from the left:

  $$A^{-1}B^{-1}\vec{y} = A^{-1}A\vec{x} = \vec{x}.$$

This computation shows that the linear transformation

$$\vec{y} = BA\vec{x}$$

is invertible and that its inverse is

$$\vec{x} = A^{-1}B^{-1}\vec{y}.$$

Theorem 2.4.7

The inverse of a product of matrices

If A and B are invertible $n \times n$ matrices, then BA is invertible as well, ar

$$(BA)^{-1} = A^{-1}B^{-1}.$$

Pay attention to the order of the matrices. (Order matters!)

To verify this result, we can multiply $A^{-1}B^{-1}$ by BA (in either orc check that the result is I_n:

$$BAA^{-1}B^{-1} = BI_nB^{-1} = BB^{-1} = I_n, \text{ and}$$
$$A^{-1}B^{-1}BA = A^{-1}A = I_n.$$

Everything works out!

To understand the order of the factors in the formula $(BA)^{-1} = A^{-1}E$ about our French coast guard story again.

To recover the actual position $\vec{x}$ from the doubly encoded position $\vec{z}$ apply the decoding transformation $\vec{y} = B^{-1}\vec{z}$ and *then* the decoding trans: $\vec{x} = A^{-1}\vec{y}$. The inverse of $\vec{z} = BA\vec{x}$ is therefore $\vec{x} = A^{-1}B^{-1}\vec{z}$, as illu Figure 3.

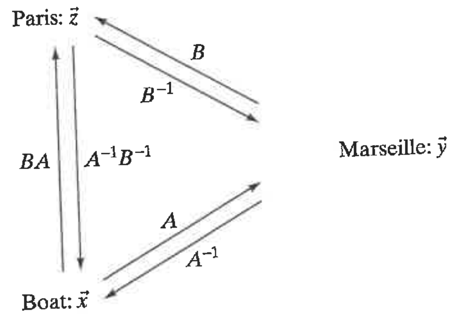


Figure 3

The following result is often useful in finding inverses:

Theorem 2.4.8

A criterion for invertibility

Let A and B be two $n \times n$ matrices such that

$$BA = I_n.$$

Then

a. A and B are both invertible,

b. $A^{-1} = B$ and $B^{-1} = A$, and

c. $AB = I_n$.

It follows from the definition of an invertible function that if $AB = I_n$ *and* $BA = I_n$, then A and B are inverses; that is, $A = B^{-1}$ and $B = A^{-1}$. Theorem 2.4.8 makes the point that the equation $BA = I_n$ *alone* guarantees that A and B are inverses. Exercise 107 illustrates the significance of this claim.

Proof To demonstrate that A is invertible, it suffices to show that the linear system $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$ (by Theorem 2.4.4b). If we multiply the equation $A\vec{x} = \vec{0}$ by B from the left, we find that $BA\vec{x} = B\vec{0} = \vec{0}$. It follows that $\vec{x} = I_n\vec{x} = BA\vec{x} = \vec{0}$, as claimed. Therefore, A is invertible. If we multiply the equation $BA = I_n$ by A^{-1} from the right, we find that $B = A^{-1}$. Matrix B, being the inverse of A, is itself invertible, and $B^{-1} = (A^{-1})^{-1} = A$. See Definition 2.4.1. Finally, $AB = AA^{-1} = I_n$.

You can use Theorem 2.4.8 to check your work when computing the inverse of a matrix. Earlier in this section we claimed that

$$B = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \quad \text{is the inverse of} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}.$$

Let's use Theorem 2.4.8b to check our work:

$$BA = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

■

EXAMPLE 2

Suppose A, B, and C are three $n \times n$ matrices such that $ABC = I_n$. Show that B is invertible, and express B^{-1} in terms of A and C.

Solution

Write $ABC = (AB)C = I_n$. We have $C(AB) = I_n$, by Theorem 2.4.8c. Since matrix multiplication is associative, we can write $(CA)B = I_n$. Applying Theorem 2.4.8 again, we conclude that B is invertible, and $B^{-1} = CA$. ■

EXAMPLE 3

For an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute the product $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. When is A invertible? If so, what is A^{-1}?

Solution

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I_2.$$

If $ad - bc \neq 0$, we can write $\underbrace{\left(\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)}_{B} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} = I_2.$

It now follows from Theorem 2.4.8 that A is invertible, with $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Conversely, if A is invertible, then we can multiply the equation $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad - bc)I_2$ with A^{-1} from the right, finding $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad - bc)A^{-1}$. Since some of the scalars a, b, c, d are nonzero (being the entries of the invertible matrix A), it follows that $ad - bc \neq 0$. ■

Theorem 2.4.9

Inverse and determinant of a 2 × 2 matrix

a. The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if (and only if) $ad - bc \neq 0$.

Quantity $ad - bc$ is called the *determinant* of A, written det(

$$\det(A) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

b. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In Chapter 6 we will introduce the determinant of a square matrix o size, and we will generalize the results of Theorem 2.4.9 to $n \times n$ ma Theorems 6.2.4 and 6.3.9.

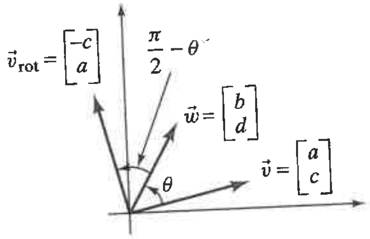


Figure 4

What is the geometrical interpretation of the determinant of a 2×2 Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and consider the column vectors $\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\vec{w}$ turns out to be helpful to introduce the auxiliary vector $\vec{v}_{\text{rot}} = \begin{bmatrix} -c \\ a \end{bmatrix}$, c rotating vector $\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix}$ through an angle of $\frac{\pi}{2}$. Let θ be the (oriented) $\vec{v}$ to $\vec{w}$, with $-\pi < \theta \leq \pi$. See Figure 4. Then

$$\det A = ad - bc \underbrace{=}_{\text{step 2}} \vec{v}_{\text{rot}} \cdot \vec{w} \underbrace{=}_{\text{step 3}} \|\vec{v}_{\text{rot}}\| \cos\left(\frac{\pi}{2} - \theta\right) \|\vec{w}\| = \|\vec{v}\| \text{ s}$$

In steps 2 and 3 we use the definition of the dot product and its geom pretation. See Definition A.4 in the Appendix.

Theorem 2.4.10

Geometrical interpretation of the determinant of a 2 × 2 matrix

If $A = [\vec{v} \quad \vec{w}]$ is a 2×2 matrix with nonzero columns $\vec{v}$ and $\vec{w}$, the

$$\det A = \det [\vec{v} \quad \vec{w}] = \|\vec{v}\| \sin\theta \|\vec{w}\|,$$

where θ is the oriented angle from $\vec{v}$ to $\vec{w}$, with $-\pi < \theta \leq \pi$. It foll

- $|\det A| = \|\vec{v}\| \, |\sin\theta| \, \|\vec{w}\|$ is the *area of the parallelogram* sp and $\vec{w}$. See Figure 5,
- $\det A = 0$ if $\vec{v}$ and $\vec{w}$ are *parallel*, meaning that $\theta = 0$ or $\theta =$
- $\det A > 0$ if $0 < \theta < \pi$, and
- $\det A < 0$ if $-\pi < \theta < 0$.

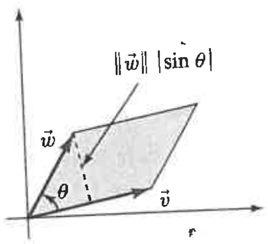


Figure 5

In Chapter 6 we will go a step further and interpret $\det A$ in terr transformation $T(\vec{x}) = A\vec{x}$.

EXAMPLE 4 Is the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ invertible? If so, find the inverse. Interpret $\det A$ geometrically.

Solution

We find the determinant $\det(A) = 1 \cdot 1 - 3 \cdot 2 = -5 \neq 0$, so that A is indeed invertible, by Theorem 2.4.9a. Then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(-5)}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix},$$

by Theorem 2.4.9b.

Furthermore, $|\det A| = 5$ is the area of the shaded parallelogram in Figure 6, and $\det A$ is negative since the angle θ from $\vec{v}$ to $\vec{w}$ is negative. ■

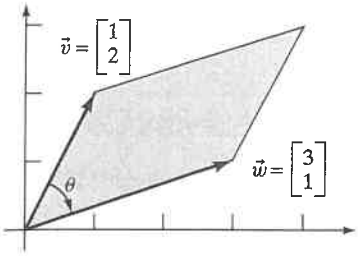


Figure 6

EXAMPLE 5 For which values of the constant k is the matrix $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$ invertible?

Solution

By Theorem 2.4.9a, the matrix A fails to be invertible if $\det A = 0$. Now

$$\det A = \det\begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix} = (1-k)(3-k) - 2 \cdot 4$$
$$= k^2 - 4k - 5 = (k-5)(k+1) = 0$$

when $k = 5$ or $k = -1$. Thus, A is invertible for all values of k *except* $k = 5$ and $k = -1$. ■

EXAMPLE 6 Consider a matrix A that represents the reflection about a line L in the plane. Use the determinant to verify that A is invertible. Find A^{-1}. Explain your answer conceptually, and interpret the determinant geometrically.

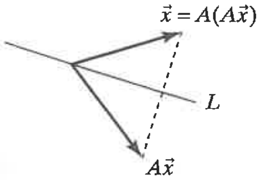


Figure 7

Solution

By Definition 2.2.2, a reflection matrix is of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Now $\det A = \det\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = -a^2 - b^2 = -1$. It turns out that A is invertible, and $A^{-1} = \frac{1}{(-1)}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = A$. It makes good sense that A is its own inverse, since $A(A\vec{x}) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$, by definition of a reflection. See Figure 7.

To interpret the determinant geometrically, recall that $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = A\vec{e}_1$ and $\vec{w} = \begin{bmatrix} b \\ -a \end{bmatrix} = A\vec{e}_2$. The parallelogram spanned by $\vec{v}$ and $\vec{w}$ is actually a unit square, with area $1 = |\det A|$, and θ is $-\frac{\pi}{2}$ since the reflection about L reverses the orientation of an angle. See Figure 8. ■

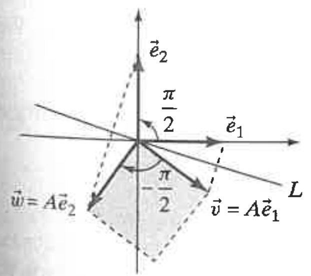


Figure 8

The Inverse of a Block Matrix (Optional)

We will conclude this chapter with two examples involving block matrice
fresh your memory, take another look at Theorem 2.3.9.

EXAMPLE 7 Let A be a block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix, and A_{12} is an $n \times$

a. For which choices of A_{11}, A_{12}, and A_{22} is A invertible?

b. If A is invertible, what is A^{-1} (in terms of A_{11}, A_{12}, A_{22})?

Solution

We are looking for an $(n+m) \times (n+m)$ matrix B such that

$$BA = I_{n+m} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}.$$

Let us partition B in the same way as A:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where B_{11} is $n \times n$, B_{22} is $m \times m$, and so on. The fact that B is the ir
means that

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix},$$

or, using Theorem 2.3.9,

$$\left| \begin{aligned} B_{11}A_{11} &= I_n \\ B_{11}A_{12} + B_{12}A_{22} &= 0 \\ B_{21}A_{11} &= 0 \\ B_{21}A_{12} + B_{22}A_{22} &= I_m \end{aligned} \right|.$$

We have to solve for the blocks B_{ij}. Applying Theorem 2.4.8 to
$B_{11}A_{11} = I_n$, we find that A_{11} is invertible, and $B_{11} = A_{11}^{-1}$. Equ
implies that $B_{21} = 0A_{11}^{-1} = 0$. Next, Equation 4 simplifies to $B_{22}A$
Theorem 2.4.8, A_{22} is invertible, and $B_{22} = A_{22}^{-1}$. Lastly, Equatio
$A_{11}^{-1}A_{12} + B_{12}A_{22} = 0$, or $B_{12}A_{22} = -A_{11}^{-1}A_{12}$, or $B_{12} = -A_{11}^{-1}$
conclude that

a. A is invertible if (and only if) both A_{11} and A_{22} are invertible
is imposed on A_{12}), and

b. If A is invertible, then its inverse is

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

Verify this result for the following example:

EXAMPLE 8

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]^{-1} = \left[\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

■

EXERCISES 2.4

GOAL *Apply the concept of an invertible function. Determine whether a matrix (or a linear transformation) is invertible, and find the inverse if it exists.*

Decide whether the matrices in Exercises 1 through 15 are invertible. If they are, find the inverse. Do the computations with paper and pencil. Show all your work.

1. $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{bmatrix}$

Decide whether the linear transformations in Exercises 16 through 20 are invertible. Find the inverse transformation if it exists. Do the computations with paper and pencil. Show all your work.

16. $\begin{aligned} y_1 &= 3x_1 + 5x_2 \\ y_2 &= 5x_1 + 8x_2 \end{aligned}$

17. $\begin{aligned} y_1 &= x_1 + 2x_2 \\ y_2 &= 4x_1 + 8x_2 \end{aligned}$

18. $\begin{aligned} y_1 &= x_2 \\ y_2 &= x_3 \\ y_3 &= x_1 \end{aligned}$

19. $\begin{aligned} y_1 &= x_1 + x_2 + x_3 \\ y_2 &= x_1 + 2x_2 + 3x_3 \\ y_3 &= x_1 + 4x_2 + 9x_3 \end{aligned}$

20. $\begin{aligned} y_1 &= x_1 + 3x_2 + 3x_3 \\ y_2 &= x_1 + 4x_2 + 8x_3 \\ y_3 &= 2x_1 + 7x_2 + 12x_3 \end{aligned}$

Which of the functions f from $\mathbb{R}$ to $\mathbb{R}$ in Exercises 21 through 24 are invertible?

21. $f(x) = x^2$

22. $f(x) = 2^x$

23. $f(x) = x^3 + x$

24. $f(x) = x^3 - x$

Which of the (nonlinear) transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ in Exercises 25 through 27 are invertible? Find the inverse if it exists.

25. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^3 \\ x_2 \end{bmatrix}$

26. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1^3 + x_2 \end{bmatrix}$

27. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 \cdot x_2 \end{bmatrix}$

28. Find the inverse of the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 22 \\ -16 \\ 8 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 13 \\ -3 \\ 9 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ -2 \\ 7 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

from $\mathbb{R}^4$ to $\mathbb{R}^4$.

29. For which values of the constant k is the following matrix invertible?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

30. For which values of the constants b and c is the following matrix invertible?

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

31. For which values of the constants a, b, and c is the following matrix invertible?

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

32. Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$ and $A^{-1} = A$.

33. Consider the matrices of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where a and b are arbitrary constants. For which values of a and b is $A^{-1} = A$?

34. Consider the diagonal matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

a. For which values of a, b, and c is A invertible? If it is invertible, what is A^{-1}?

b. For which values of the diagonal elements is a diagonal matrix (of arbitrary size) invertible?

35. a. Consider the upper triangular 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

For which values of a, b, c, d, e, and f is A invertible?

b. More generally, when is an upper triangular matrix (of arbitrary size) invertible?

c. If an upper triangular matrix is invertible, is its inverse an upper triangular matrix as well?

d. When is a lower triangular matrix invertible?

36. To determine whether a square matrix A is invertible, it is not always necessary to bring it into reduced row-echelon form. Instead, reduce A to (upper or lower) triangular form, using elementary row operations. Show that A is invertible if (and only if) all entries on the diagonal of this triangular form are nonzero.

37. If A is an invertible matrix and c is a nonzero scalar, is the matrix cA invertible? If so, what is the relationship between A^{-1} and $(cA)^{-1}$?

38. Find A^{-1} for $A = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$.

39. Consider a square matrix that differs from t
matrix at just one entry, off the diagonal, for

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

In general, is a matrix M of this form inve
what is the M^{-1}?

40. Show that if a square matrix A has two equ
then A is not invertible.

41. Which of the following linear transformat
$\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Find the inverse if

a. Reflection about a plane

b. Orthogonal projection onto a plane

c. Scaling by a factor of 5 [i.e., $T(\vec{v}) = 5\vec{v}$
tors $\vec{v}$]

d. Rotation about an axis

42. A square matrix is called a *permutation ma*
tains a 1 exactly once in each row and in
with all other entries being 0. Examples a

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Are permutation matrices invertible? If so
a permutation matrix as well?

43. Consider two invertible $n \times n$ matrices A
linear transformation $\vec{y} = A(B\vec{x})$ invertib
is the inverse? *Hint*: Solve the equation $\vec{y}$
for $B\vec{x}$ and then for $\vec{x}$.

44. Consider the $n \times n$ matrix M_n, with $n \geq$
all integers $1, 2, 3, \ldots, n^2$ as its entries
quence, column by column; for example

$$M_4 = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

a. Determine the rank of M_4.

b. Determine the rank of M_n.

c. For which n is M_n invertible?

45. To gauge the complexity of a computati
ematicians and computer scientists cou
elementary operations (additions, sub
plications, and divisions) required. Fo
we will sometimes consider multiplic
sions only, referring to those jointly
operations. As an example, we exami
inverting a 2×2 matrix by eliminatio

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \div a, \text{ requires 2 multiplicative operations: } b/a \text{ and } 1/a$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & b' & e & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \begin{array}{l}(\text{where } b' = b/a, \text{ and } e = 1/a) \\ -c\ (\text{I}), \text{ requires 2 multiplicative operations: } cb' \text{ and } ce\end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & b' & e & 0 \\ 0 & d' & g & 1 \end{array}\right] \quad \div d', \text{ requires 2 multiplicative operations}$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & b' & e & 0 \\ 0 & 1 & g' & h \end{array}\right] \quad -b'\ (\text{II}), \text{ requires 2 multiplicative operations}$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & e' & f \\ 0 & 1 & g' & h \end{array}\right]$$

The whole process requires eight multiplicative operations. Note that we do not count operations with predictable results, such as $1a$, $0a$, a/a, $0/a$.

a. How many multiplicative operations are required to invert a 3×3 matrix by elimination?

b. How many multiplicative operations are required to invert an $n \times n$ matrix by elimination?

c. If it takes a slow hand-held calculator 1 second to invert a 3×3 matrix, how long will it take the same calculator to invert a 12×12 matrix? Assume that the matrices are inverted by Gauss–Jordan elimination and that the duration of the computation is proportional to the number of multiplications and divisions involved.

46. Consider the linear system

$$A\vec{x} = \vec{b},$$

where A is an invertible matrix. We can solve this system in two different ways:

- By finding the reduced row-echelon form of the augmented matrix $\left[A \mid \vec{b}\right]$,
- By computing A^{-1} and using the formula $\vec{x} = A^{-1}\vec{b}$.

In general, which approach requires fewer multiplicative operations? See Exercise 45.

47. Give an example of a noninvertible function f from $\mathbb{R}$ to $\mathbb{R}$ and a number b such that the equation

$$f(x) = b$$

has a unique solution.

48. Consider an invertible linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$, with inverse $L = T^{-1}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. In Exercise 2.2.29 we show that L is a linear transformation, so that $L(\vec{y}) = B\vec{y}$ for some $m \times n$ matrix B. Use the equations $BA = I_n$ and $AB = I_m$ to show that $n = m$. *Hint:* Think about the number of solutions of the linear systems $A\vec{x} = \vec{0}$ and $B\vec{y} = \vec{0}$.

49. *Input–Output Analysis.* (This exercise builds on Exercises 1.1.24, 1.2.39, 1.2.40, and 1.2.41). Consider the industries $J_1, J_2, \dots, J_n$ in an economy. Suppose the consumer demand vector is $\vec{b}$, the output vector is $\vec{x}$, and the demand vector of the jth industry is $\vec{v}_j$. (The ith component a_{ij} of $\vec{v}_j$ is the demand industry J_j puts on industry J_i, per unit of output of J_j.) As we have seen in Exercise 1.2.40, the output $\vec{x}$ just meets the aggregate demand if

$$\underbrace{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b}}_{\text{aggregate demand}} = \underbrace{\vec{x}}_{\text{output}}.$$

This equation can be written more succinctly as

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \vec{b} = \vec{x},$$

or $A\vec{x} + \vec{b} = \vec{x}$. The matrix A is called the *technology matrix* of this economy; its coefficients a_{ij} describe the interindustry demand, which depends on the technology used in the production process. The equation

$$A\vec{x} + \vec{b} = \vec{x}$$

describes a linear system, which we can write in the customary form:

$$\begin{aligned} \vec{x} - A\vec{x} &= \vec{b} \\ I_n\vec{x} - A\vec{x} &= \vec{b} \\ (I_n - A)\vec{x} &= \vec{b}. \end{aligned}$$

If we want to know the output $\vec{x}$ required to satisfy a given consumer demand $\vec{b}$ (this was our objective in the previous exercises), we can solve this linear system, preferably via the augmented matrix.

In economics, however, we often ask other questions: If $\vec{b}$ *changes*, how will $\vec{x}$ change in response? If the consumer demand on one industry increases by 1 unit and the consumer demand on the other industries remains unchanged, how will $\vec{x}$ change?[11] If we

[11] The relevance of questions like these became particularly clear during World War II, when the demand on certain industries suddenly changed dramatically. When U.S. President F. D. Roosevelt asked for 50,000 airplanes to be built, it was easy enough to predict that the country would have to produce more aluminum. Unexpectedly, the demand for copper dramatically increased (why?). A copper shortage then occurred, which was solved by borrowing silver from Fort Knox. People realized that input–output analysis can be effective in modeling and predicting chains of increased demand like this. After World War II, this technique rapidly gained acceptance and was soon used to model the economies of more than 50 countries.

ask questions like these, we think of the output $\vec{x}$ as a *function* of the consumer demand $\vec{b}$.

If the matrix $(I_n - A)$ is invertible,[12] we can express $\vec{x}$ as a function of $\vec{b}$ (in fact, as a linear transformation):

$$\vec{x} = (I_n - A)^{-1}\vec{b}.$$

a. Consider the example of the economy of Israel in 1958 (discussed in Exercise 1.2.41). Find the technology matrix A, the matrix $(I_n - A)$, and its inverse $(I_n - A)^{-1}$.

b. In the example discussed in part (a), suppose the consumer demand on agriculture (Industry 1) is 1 unit (1 million pounds), and the demands on the other two industries are zero. What output $\vec{x}$ is required in this case? How does your answer relate to the matrix $(I_n - A)^{-1}$?

c. Explain, in terms of economics, why the diagonal elements of the matrix $(I_n - A)^{-1}$ you found in part (a) must be at least 1.

d. If the consumer demand on manufacturing increases by 1 (from whatever it was), and the consumer demand on the other two industries remains the same, how will the output have to change? How does your answer relate to the matrix $(I_n - A)^{-1}$?

e. Using your answers in parts (a) through (d) as a guide, explain in general (not just for this example) what the columns and the entries of the matrix $(I_n - A)^{-1}$ tell you, in terms of economics. Those who have studied multivariable calculus may wish to consider the partial derivatives

$$\frac{\partial x_i}{\partial b_j}.$$

50. This exercise refers to Exercise 49a. Consider the entry $k = a_{11} = 0.293$ of the technology matrix A. Verify that the entry in the first row and the first column of $(I_n - A)^{-1}$ is the value of the geometrical series $1 + k + k^2 + \cdots$. Interpret this observation in terms of economics.

51. a. Consider an $n \times m$ matrix A with $\text{rank}(A) < n$. Show that there exists a vector $\vec{b}$ in $\mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ is inconsistent. *Hint*: For $E = \text{rref}(A)$, show that there exists a vector $\vec{c}$ in $\mathbb{R}^n$ such that the system $E\vec{x} = \vec{c}$ is inconsistent; then, "work backward."

b. Consider an $n \times m$ matrix A with $n > m$. Show that there exists a vector $\vec{b}$ in $\mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

52. For

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{bmatrix},$$

find a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x}$ … inconsistent. See Exercise 51.

53. Let $A = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$ in all parts of this problem.

a. Find a scalar λ (lambda) such that the matri… fails to be invertible. There are two solution… one and use it in parts (b) and (c).

b. For the λ you chose in part (a), find th… $A - \lambda I_2$; then find a nonzero vector $\vec{x}$ … $(A - \lambda I_2)\vec{x} = \vec{0}$. (This can be done, since … fails to be invertible.)

c. Note that the equation $(A - \lambda I_2)\vec{x} = \vec{0}$ ca… ten as $A\vec{x} - \lambda\vec{x} = \vec{0}$, or $A\vec{x} = \lambda\vec{x}$. Chec… equation $A\vec{x} = \lambda\vec{x}$ holds for your λ fro… and your $\vec{x}$ from part (b).

54. Let $A = \begin{bmatrix} 1 & 10 \\ -3 & 12 \end{bmatrix}$. Using Exercise 53 as a … a scalar λ and a nonzero vector $\vec{x}$ such that A…

In Exercises 55 through 65, show that the gi… A is invertible, and find the inverse. Interpret … transformation $T(\vec{x}) = A\vec{x}$ and the inverse tr… tion $T^{-1}(\vec{y}) = A^{-1}\vec{y}$ geometrically. Interpret … metrically. In your figure, show the angle θ and … $\vec{v}$ and $\vec{w}$ introduced in Theorem 2.4.10.

55. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

56. $\begin{bmatrix} \cos\alpha & -\sin… \\ \sin\alpha & \cos… \end{bmatrix}$

57. $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$

58. $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

59. $\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$

60. $\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$

61. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

62. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

63. $\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

64. $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

65. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

66. Consider two $n \times n$ matrices A and B … product AB is invertible. Show that th… and B are both invertible. *Hint*: $AB(AB$… and $(AB)^{-1}AB = I_n$. Use Theorem 2.4.8.

For two invertible n × n matrices A and … which of the formulas stated in Exercises … are necessarily true.

67. $(A + B)^2 = A^2 + 2AB + B^2$

68. $(A - B)(A + B) = A^2 - B^2$

69. $A + B$ is invertible, and $(A + B)^{-1} = …$

70. A^2 is invertible, and $(A^2)^{-1} = (A^{-1})^2$

71. $ABB^{-1}A^{-1} = I_n$

[12] This will always be the case for a "productive" economy. See Exercise 103.

72. $ABA^{-1} = B$

73. $(ABA^{-1})^3 = AB^3A^{-1}$

74. $(I_n + A)(I_n + A^{-1}) = 2I_n + A + A^{-1}$

75. $A^{-1}B$ is invertible, and $(A^{-1}B)^{-1} = B^{-1}A$

76. Find all linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that

$$T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}.$$

Hint: We are looking for the 2×2 matrices A such that

$$A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}.$$

These two equations can be combined to form the matrix equation

$$A\begin{bmatrix}1 & 2\\2 & 5\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}.$$

77. Using the last exercise as a guide, justify the following statement:
Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be vectors in $\mathbb{R}^m$ such that the matrix

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

is invertible. Let $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ be arbitrary vectors in $\mathbb{R}^n$. Then there exists a unique linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that $T(\vec{v}_i) = \vec{w}_i$, for all $i = 1, \dots, m$. Find the matrix A of this transformation in terms of S and

$$B = \begin{bmatrix} | & | & & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \\ | & | & & | \end{bmatrix}.$$

78. Find the matrix A of the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ with

$$T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}7\\5\\3\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

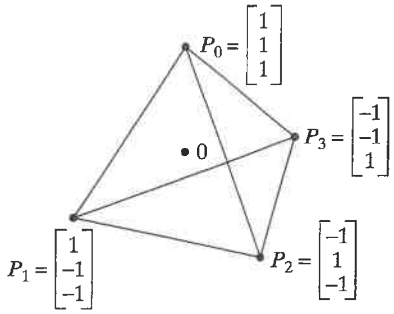
Compare with Exercise 77.

79. Find the matrix A of the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ with

$$T\begin{bmatrix}3\\1\end{bmatrix} = 2\begin{bmatrix}3\\1\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}1\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}.$$

Compare with Exercise 77.

80. Consider the regular tetrahedron sketched below, whose center is at the origin.



Let T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the rotation about the axis through the points 0 and P_2 that transforms P_1 into P_3. Find the images of the four corners of the tetrahedron under this transformation.

$$P_0 \xrightarrow{T}$$
$$P_1 \to P_3$$
$$P_2 \to$$
$$P_3 \to$$

Let L from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection about the plane through the points 0, P_0, and P_3. Find the images of the four corners of the tetrahedron under this transformation.

$$P_0 \xrightarrow{L}$$
$$P_1 \to$$
$$P_2 \to$$
$$P_3 \to$$

Describe the transformations in parts (a) through (c) geometrically.

a. T^{-1} **b.** L^{-1}

c. $T^2 = T \circ T$ (the composite of T with itself)

d. Find the images of the four corners under the transformations $T \circ L$ and $L \circ T$. Are the two transformations the same?

$$P_0 \xrightarrow{T \circ L} \qquad P_0 \xrightarrow{L \circ T}$$
$$P_1 \to \qquad P_1 \to$$
$$P_2 \to \qquad P_2 \to$$
$$P_3 \to \qquad P_3 \to$$

e. Find the images of the four corners under the transformation $L \circ T \circ L$. Describe this transformation geometrically.

81. Find the matrices of the transformations T and L defined in Exercise 80.

82. Consider the matrix

$$E = \begin{bmatrix}1 & 0 & 0\\-3 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

and an arbitrary 3×3 matrix

$$A = \begin{bmatrix}a & b & c\\d & e & f\\g & h & k\end{bmatrix}.$$

a. Compute EA. Comment on the relationship between A and EA, in terms of the technique of elimination we learned in Section 1.2.

b. Consider the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and an arbitrary 3×3 matrix A. Compute EA. Comment on the relationship between A and EA.

c. Can you think of a 3×3 matrix E such that EA is obtained from A by swapping the last two rows (for any 3×3 matrix A)?

d. The matrices of the forms introduced in parts (a), (b), and (c) are called *elementary*: An $n \times n$ matrix E is elementary if it can be obtained from I_n by performing one of the three elementary row operations on I_n. Describe the format of the three types of elementary matrices.

83. Are elementary matrices invertible? If so, is the inverse of an elementary matrix elementary as well? Explain the significance of your answers in terms of elementary row operations.

84. a. Justify the following: If A is an $n \times m$ matrix, then there exist elementary $n \times n$ matrices $E_1, E_2, \dots, E_p$ such that

$$\text{rref}(A) = E_1 E_2 \cdots E_p A.$$

b. Find such elementary matrices $E_1, E_2, \dots, E_p$ for

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}.$$

85. a. Justify the following: If A is an $n \times m$ matrix, then there exists an invertible $n \times n$ matrix S such that

$$\text{rref}(A) = SA.$$

b. Find such an invertible matrix S for

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}.$$

86. a. Justify the following: Any invertible matrix is a product of elementary matrices.

b. Write $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices.

87. Write all possible forms of elementary 2×2 matrices E. In each case, describe the transformation $\vec{y} = E\vec{x}$ geometrically.

88. Consider an invertible $n \times n$ matrix A and an $n \times n$ matrix B. A certain sequence of elementary row operations transforms A into I_n.

a. What do you get when you apply the same row operations in the same order to the matrix AB?

b. What do you get when you apply the same [op]erations to I_n?

89. Is the product of two lower triangular matrice[s] [a lower] triangular matrix as well? Explain your answe[r]

90. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}.$$

a. Find lower triangular elementary mat[rices] $E_2, \dots, E_m$ such that the product

$$E_m \cdots E_2 E_1 A$$

is an upper triangular matrix U. *Hint*: Us[e elemen]tary row operations to eliminate the ent[ries below] the diagonal of A.

b. Find lower triangular elementary ma[trices] $M_2, \dots, M_m$ and an upper triangular ma[trix U such] that

$$A = M_1 M_2 \cdots M_m U.$$

c. Find a lower triangular matrix L and an [upper trian]gular matrix U such that

$$A = LU.$$

Such a representation of an invertibl[e matrix is] called an *LU-factorization*. The meth[od outlined] in this exercise to find an LU-factoriz[ation can be] streamlined somewhat, but we have s[een the ma]jor ideas. An LU-factorization (as intr[oduced here)] does not always exist. See Exercise 92.

d. Find a lower triangular matrix L with [1's on the di]agonal, an upper triangular matrix U [with 1's on] the diagonal, and a diagonal matrix [D such that] $A = LDU$. Such a representation of [an invertible] matrix is called an *LDU-factorization*

91. Knowing an LU-factorization of a matri[x ... makes it] much easier to solve a linear system

$$A\vec{x} = \vec{b}.$$

Consider the LU-factorization

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -3 & -5 & 6 & -5 \\ 1 & 4 & 6 & 20 \\ -1 & 6 & 20 & 43 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 8 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \end{bmatrix}$$

$$= LU.$$

Suppose we have to solve the system $A\vec{x} = LU\vec{x} = \vec{b}$, where

$$\vec{b} = \begin{bmatrix} -3 \\ 14 \\ 9 \\ 33 \end{bmatrix}.$$

a. Set $\vec{y} = U\vec{x}$, and solve the system $L\vec{y} = \vec{b}$, by forward substitution (finding first y_1, then y_2, etc.). Do this using paper and pencil. Show all your work.

b. Solve the system $U\vec{x} = \vec{y}$, using back substitution, to find the solution $\vec{x}$ of the system $A\vec{x} = \vec{b}$. Do this using paper and pencil. Show all your work.

$$\begin{tikzcd} & \vec{y} \arrow[dr, "L"] & \\ \vec{x} \arrow[ur, "U"] \arrow[rr, "A"'] & & \vec{b} \end{tikzcd}$$

92. Show that the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ cannot be written in the form $A = LU$, where L is lower triangular and U is upper triangular.

93. In this exercise we will examine which invertible $n \times n$ matrices A admit an LU-factorization $A = LU$, as discussed in Exercise 90. The following definition will be useful: For $m = 1, \ldots, n$, the *principal submatrix* $A^{(m)}$ of A is obtained by omitting all rows and columns of A past the mth. For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \end{bmatrix}$$

has the principal submatrices

$$A^{(1)} = [1],\ A^{(2)} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix},\ A^{(3)} = A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \end{bmatrix}.$$

We will show that an invertible $n \times n$ matrix A admits an LU-factorization $A = LU$ if (and only if) all its principal submatrices are invertible.

a. Let $A = LU$ be an LU-factorization of an $n \times n$ matrix A. Use block matrices to show that $A^{(m)} = L^{(m)}U^{(m)}$ for $m = 1, \ldots, n$.

b. Use part (a) to show that if an invertible $n \times n$ matrix A has an LU-factorization, then all its principal submatrices $A^{(m)}$ are invertible.

c. Consider an $n \times n$ matrix A whose principal submatrices are all invertible. Show that A admits an LU-factorization. *Hint*: By induction, you can assume that $A^{(n-1)}$ has an LU-factorization $A^{(n-1)} = L'U'$. Use block matrices to find an LU-factorization for A. Alternatively, you can explain this result in terms of Gauss–Jordan elimination (if all principal submatrices are invertible, then no row swaps are required).

94. a. Show that if an invertible $n \times n$ matrix A admits an LU-factorization, then it admits an LDU-factorization. See Exercise 90 d.

b. Show that if an invertible $n \times n$ matrix A admits an LDU-factorization, then this factorization is unique. *Hint*: Suppose that $A = L_1D_1U_1 = L_2D_2U_2$. Then $U_2U_1^{-1} = D_2^{-1}L_2^{-1}L_1D_1$ is diagonal (why?). Conclude that $U_2 = U_1$.

95. Consider a block matrix

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. For which choices of A_{11} and A_{22} is A invertible? In these cases, what is A^{-1}?

96. Consider a block matrix

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. For which choices of A_{11}, A_{21}, and A_{22} is A invertible? In these cases, what is A^{-1}?

97. Consider the block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & 0 & A_{23} \end{bmatrix},$$

where A_{11} is an invertible matrix. Determine the rank of A in terms of the ranks of the blocks A_{11}, A_{12}, A_{13}, and A_{23}.

98. Consider the block matrix

$$A = \begin{bmatrix} I_n & \vec{v} \\ \vec{w} & 1 \end{bmatrix},$$

where $\vec{v}$ is a vector in $\mathbb{R}^n$, and $\vec{w}$ is a row vector with n components. For which choices of $\vec{v}$ and $\vec{w}$ is A invertible? In these cases, what is A^{-1}?

99. Find all invertible $n \times n$ matrices A such that $A^2 = A$.

100. Find a nonzero $n \times n$ matrix A with identical entries such that $A^2 = A$.

101. Consider two $n \times n$ matrices A and B whose entries are positive or zero. Suppose that all entries of A are less than or equal to s, and all column sums of B are less than or equal to r (the jth column sum of a matrix is the sum of all the entries in its jth column). Show that all entries of the matrix AB are less than or equal to sr.

102. (This exercise builds on Exercise 101.) Consider an $n \times n$ matrix A whose entries are positive or zero. Suppose that all column sums of A are less than 1. Let r be the largest column sum of A.

a. Show that the entries of A^m are less than or equal to r^m, for all positive integers m.

b. Show that

$$\lim_{m \to \infty} A^m = 0$$

(meaning that all entries of A^m approach zero).

c. Show that the infinite series

$$I_n + A + A^2 + \cdots + A^m + \cdots$$

converges (entry by entry).

d. Compute the product

$$(I_n - A)(I_n + A + A^2 + \cdots + A^m).$$

Simplify the result. Then let m go to infinity, and thus show that

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots.$$

103. (This exercise builds on Exercises 49, 101, and 102.)

a. Consider the industries $J_1, \ldots, J_n$ in an economy. We say that industry J_j is *productive* if the jth column sum of the technology matrix A is less than 1. What does this mean in terms of economics?

b. We say that an economy is productive if all of its industries are productive. Exercise 102 shows that if A is the technology matrix of a productive economy, then the matrix $I_n - A$ is invertible. What does this result tell you about the ability of a productive economy to satisfy consumer demand?

c. Interpret the formula

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots$$

derived in Exercise 102d in terms of economics.

104. The color of light can be represented in a vector

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix},$$

where R = amount of red, G = amount of green, and B = amount of blue. The human eye and the brain transform the incoming signal into the signal

$$\begin{bmatrix} I \\ L \\ S \end{bmatrix},$$

where

$$\begin{aligned} \text{intensity} \quad & I = \frac{R+G+B}{3} \\ \text{long-wave signal} \quad & L = R - G \\ \text{short-wave signal} \quad & S = B - \frac{R+G}{2}. \end{aligned}$$

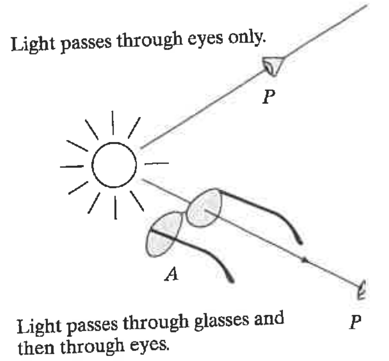
a. Find the matrix P representing the transformation from

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} I \\ L \\ S \end{bmatrix}.$$

b. Consider a pair of yellow sunglasses for wa that cuts out all blue light and passes al green light. Find the 3 × 3 matrix A that r the transformation incoming light underg passes through the sunglasses. All the entri matrix A will be 0's and 1's.

c. Find the matrix for the composite trans that light undergoes as it first passes throug glasses and then the eye.

d. As you put on the sunglasses, the signal y (intensity, long- and short-wave signals) a transformation. Find the matrix M of th mation. Feel free to use technology.



105. A village is divided into three mutual groups called *clans*. Each person in the vil to a clan, and this identification is perm are rigid rules concerning marriage: A per clan can only marry a person from one othe rules are encoded in the matrix A below. the 2–3 entry is 1 indicates that marriage be from clan III and a woman from clan II is clan of a child is determined by the mot indicated by the matrix B. According to siblings belong to the same clan.

$$\begin{array}{c} \text{Husband's clan} \\ \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array} \end{array}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \quad \text{Wife clan}$$

$$\begin{array}{c} \text{Mother's clan} \\ \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array} \end{array}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \quad \text{Chi clar}$$

The identification of a person with clan sented by the vector

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

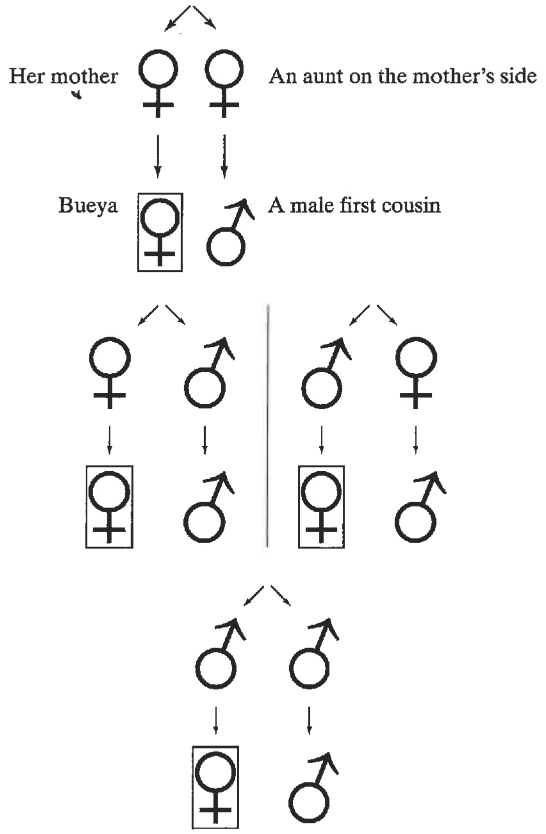
and likewise for the two other clans. Matrix A transforms the husband's clan into the wife's clan (if $\vec{x}$ represents the husband's clan, then $A\vec{x}$ represents the wife's clan).

a. Are the matrices A and B invertible? Find the inverses if they exist. What do your answers mean, in practical terms?

b. What is the meaning of B^2, in terms of the rules of the community?

c. What is the meaning of AB and BA, in terms of the rules of the community? Are AB and BA the same?

d. Bueya is a young woman who has many male first cousins, both on her mother's and on her father's sides. The kinship between Bueya and each of her male cousins can be represented by one of the four diagrams below:



In each of the four cases, find the matrix that gives you the cousin's clan in terms of Bueya's clan.

e. According to the rules of the village, could Bueya marry a first cousin? (We do not know Bueya's clan.)

106. As background to this exercise, see Exercise 45.

a. If you use Theorem 2.3.4, how many multiplications of scalars are necessary to multiply two 2×2 matrices?

b. If you use Theorem 2.3.4, how many multiplications are needed to multiply an $n \times p$ and a $p \times m$ matrix?

In 1969, the German mathematician Volker Strassen surprised the mathematical community by showing that two 2×2 matrices can be multiplied with only seven multiplications of numbers. Here is his trick: Suppose you have to find AB for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. First compute

$$\begin{aligned} h_1 &= (a+d)(p+s) \\ h_2 &= (c+d)p \\ h_3 &= a(q-s) \\ h_4 &= d(r-p) \\ h_5 &= (a+b)s \\ h_6 &= (c-a)(p+q) \\ h_7 &= (b-d)(r+s). \end{aligned}$$

Then

$$AB = \begin{bmatrix} h_1 + h_4 - h_5 + h_7 & h_3 + h_5 \\ h_2 + h_4 & h_1 + h_3 - h_2 + h_6 \end{bmatrix}.$$

107. Let $\mathbb{N}$ be the set of all positive integers, 1, 2, 3, ... We define two functions f and g from $\mathbb{N}$ to $\mathbb{N}$:

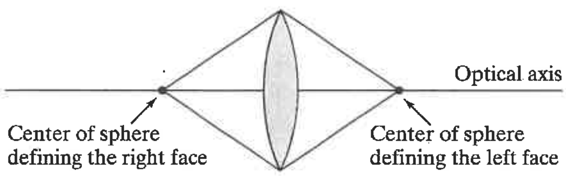
$$f(x) = 2x, \quad \text{for all } x \text{ in } \mathbb{N}$$

$$g(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

Find formulas for the composite functions $g(f(x))$ and $f(g(x))$. Is one of them the identity transformation from $\mathbb{N}$ to $\mathbb{N}$? Are the functions f and g invertible?

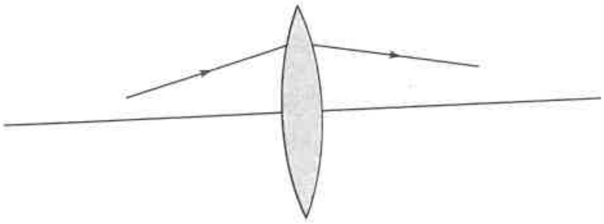
108. *Geometrical optics.* Consider a thin biconvex lens with two spherical faces.

This is a good model for the lens of the human eye and for the lenses used in many optical instruments, such as reading glasses, cameras, microscopes, and telescopes. The line through the centers of the spheres defining the two faces is called the *optical axis* of the lens.

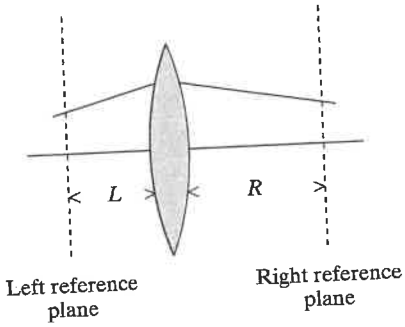


In this exercise, we learn how we can track the path of a ray of light as it passes through the lens, provided that the following conditions are satisfied:

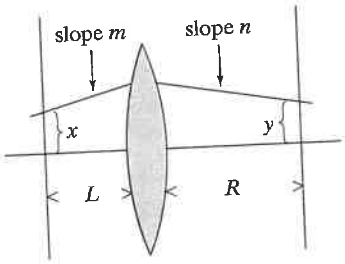
- The ray lies in a plane with the optical axis.
- The angle the ray makes with the optical axis is small.



To keep track of the ray, we introduce two *reference planes* perpendicular to the optical axis, to the left and to the right of the lens.



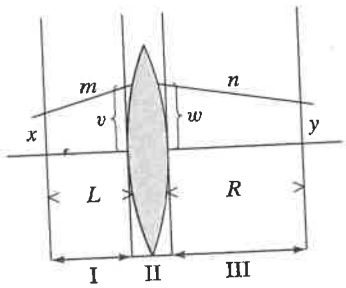
We can characterize the incoming ray by its slope m and its intercept x with the left reference plane. Likewise, we characterize the outgoing ray by slope n and intercept y.



We want to know how the outgoing ray depends on the incoming ray; that is, we are interested in the transformation

$$T: \mathbb{R}^2 \to \mathbb{R}^2; \quad \begin{bmatrix} x \\ m \end{bmatrix} \to \begin{bmatrix} y \\ n \end{bmatrix}.$$

We will see that T can be approximated by a linear transformation provided that m is small, as we assumed. To study this transformation, we divide the path of the ray into three segments, as shown in the following figure:



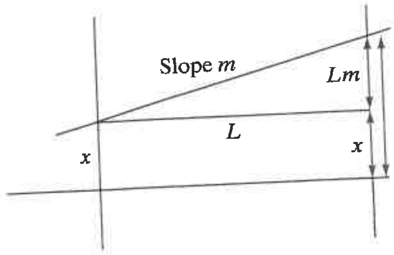
We have introduced two auxiliary reference p
rectly to the left and to the right of the lens. C
formation

$$\begin{bmatrix} x \\ m \end{bmatrix} \to \begin{bmatrix} y \\ n \end{bmatrix}$$

can now be represented as the composite of
pler transformations:

$$\begin{bmatrix} x \\ m \end{bmatrix} \to \begin{bmatrix} v \\ m \end{bmatrix} \to \begin{bmatrix} w \\ n \end{bmatrix} \to \begin{bmatrix} y \\ n \end{bmatrix}.$$

From the definition of the slope of a line,
relations $v = x + Lm$ and $y = w + Rn$.



$$\begin{bmatrix} v \\ m \end{bmatrix} = \begin{bmatrix} x + Lm \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \Big[$$

$$\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ n \end{bmatrix}$$

$$\begin{bmatrix} x \\ m \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}]{} \begin{bmatrix} v \\ m \end{bmatrix} \xrightarrow{\qquad} \begin{bmatrix} w \\ n \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 \\ 0 \end{bmatrix}]{}$$

It would lead us too far into physics to de
for the transformation

$$\begin{bmatrix} v \\ m \end{bmatrix} \to \begin{bmatrix} w \\ n \end{bmatrix}$$

here.[13] Under the assumptions we have m
formation is well approximated by

$$\begin{bmatrix} w \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} v \\ m \end{bmatrix},$$

for some positive constant k (this form
$w = v$).

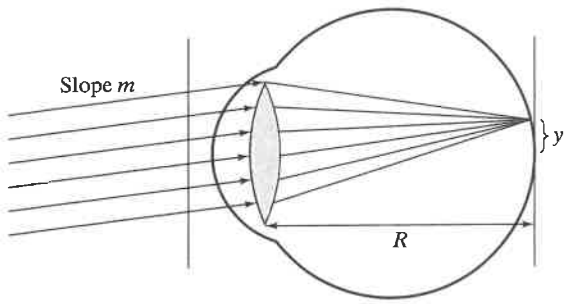
$$\begin{bmatrix} x \\ m \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}]{} \begin{bmatrix} v \\ m \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}]{} \begin{bmatrix} w \\ n \end{bmatrix}$$

[13] See, for example, Paul Bamberg and Shlo
Course in Mathematics for Students of Phy
University Press, 1991.

The transformation $\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix}$ is represented by the matrix product

$$\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - Rk & L + R - kLR \\ -k & 1 - kL \end{bmatrix}.$$

a. *Focusing parallel rays.* Consider the lens in the human eye, with the retina as the right reference plane. In an adult, the distance R is about 0.025 meters (about 1 inch). The ciliary muscles allow you to vary the shape of the lens and thus the lens constant k, within a certain range. What value of k enables you to focus parallel incoming rays, as shown in the figure? This value of k will allow you to see a distant object clearly. (The customary unit of measurement for k is 1 diopter $= \frac{1}{1 \text{ meter}}$.)

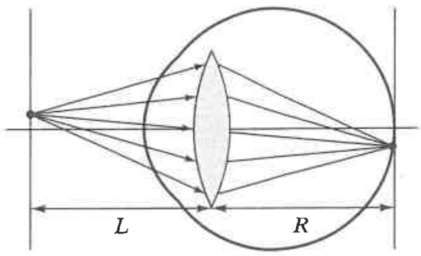


Hint: In terms of the transformation

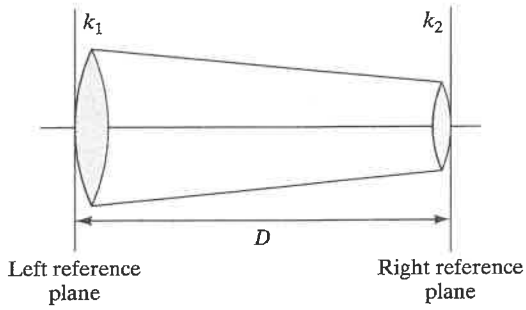
$$\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix},$$

you want y to be independent of x (y must depend on the slope m alone). Explain why $1/k$ is called the *focal length* of the lens.

b. What value of k enables you to read this text from a distance of $L = 0.3$ meters? Consider the following figure (which is not to scale).



c. *The telescope.* An astronomical telescope consists of two lenses with the same optical axis.



Find the matrix of the transformation

$$\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix},$$

in terms of k_1, k_2, and D. For given values of k_1 and k_2, how do you choose D so that parallel incoming rays are converted into parallel outgoing rays? What is the relationship between D and the focal lengths of the two lenses, $1/k_1$ and $1/k_2$?

Chapter Two Exercises

TRUE OR FALSE?

1. The matrix $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$ represents a rotation combined with a scaling.
2. If A is any invertible $n \times n$ matrix, then A commutes with A^{-1}.
3. The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$ is a linear transformation.
4. Matrix $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ represents a rotation.
5. If A is any invertible $n \times n$ matrix, then $\text{rref}(A) = I_n$.
6. The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.
7. The formula $AB = BA$ holds for all $n \times n$ matrices A and B.
8. If $AB = I_n$ for two $n \times n$ matrices A and B, then A must be the inverse of B.
9. If A is a 3×4 matrix and B is a 4×5 matrix, then AB will be a 5×3 matrix.
10. The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.

11. Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.

12. There exists a real number k such that the matrix $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$ fails to be invertible.

13. There exists a real number k such that the matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ fails to be invertible.

14. $A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$ is a regular transition matrix.

15. The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A.

16. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

17. Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.

18. Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.

19. There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

20. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x-3)^2 - (x+3)^2 \end{bmatrix}$ is a linear transformation.

21. There exists an invertible $n \times n$ matrix with two identical rows.

22. If $A^2 = I_n$, then matrix A must be invertible.

23. There exists a matrix A such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

24. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

25. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents a reflection about a line.

26. For every regular transition matrix A there exists a transition matrix B such that $AB = B$.

27. The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is always a scalar multiple of I_2.

28. There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

29. There exists a positive integer n such that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2$.

30. There exists an invertible 2×2 matrix A s $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

31. There exists a regular transition matrix A of s such that $A^2 = A$.

32. If A is any transition matrix and B is any posi sition matrix, then AB must be a positive trans trix.

33. If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, the $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible as well.

34. If A^2 is invertible, then matrix A itself must ible.

35. If $A^{17} = I_2$, then matrix A must be I_2.

36. If $A^2 = I_2$, then matrix A must be either I_2

37. If matrix A is invertible, then matrix $5A$ mus ible as well.

38. If A and B are two 4×3 matrices such tha for all vectors $\vec{v}$ in $\mathbb{R}^3$, then matrices A and equal.

39. If matrices A and B commute, then the forr BA^2 must hold.

40. If $A^2 = A$ for an invertible $n \times n$ matrix A, be I_n.

41. If A is any transition matrix such that A^{10} then A^{101} must be positive as well.

42. If a transition matrix A is invertible, then a transition matrix as well.

43. If matrices A and B are both invertible $A + B$ must be invertible as well.

44. The equation $A^2 = A$ holds for all $2 \times$ representing a projection.

45. The equation $A^{-1} = A$ holds for all $2 \times$ representing a reflection.

46. The formula $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ holds fc 2×2 matrices A and for all vectors $\vec{v}$ an

47. There exist a 2×3 matrix A and a 3×2 that $AB = I_2$.

48. There exist a 3×2 matrix A and a 2×3 that $AB = I_3$.

49. If $A^2 + 3A + 4I_3 = 0$ for a 3×3 matrix be invertible.

50. If A is an $n \times n$ matrix such that $A^2 =$ $I_n + A$ must be invertible.

51. If matrix A commutes with B, and B commutes with C, then matrix A must commute with C.

52. If T is any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, then $T(\vec{v} \times \vec{w}) = T(\vec{v}) \times T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$.

53. There exists an invertible 10×10 matrix that has 92 ones among its entries.

54. The formula $\operatorname{rref}(AB) = \operatorname{rref}(A)\operatorname{rref}(B)$ holds for all $n \times p$ matrices A and for all $p \times m$ matrices B.

55. There exists an invertible matrix S such that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$ is a diagonal matrix.

56. If the linear system $A^2\vec{x} = \vec{b}$ is consistent, then the system $A\vec{x} = \vec{b}$ must be consistent as well.

57. There exists an invertible 2×2 matrix A such that $A^{-1} = -A$.

58. There exists an invertible 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

59. If a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the orthogonal projection onto a line L, then the equation $a^2 + b^2 + c^2 + d^2 = 1$ must hold.

60. If A is an invertible 2×2 matrix and B is any 2×2 matrix, then the formula $\operatorname{rref}(AB) = \operatorname{rref}(B)$ must hold.

61. There is a transition matrix A such that $\lim\limits_{m \to \infty} A^m$ fails to exist.

62. For every transition matrix A there exists a nonzero vector $\vec{x}$ such that $A\vec{x} = \vec{x}$.