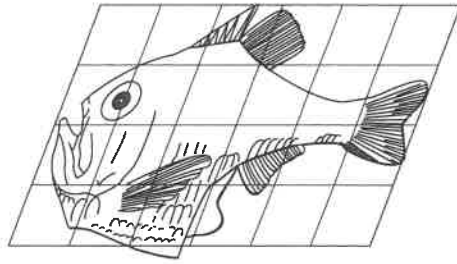


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# Linear Transformations

## 2.1 Introduction to Linear Transformations and Their Inverses

Imagine yourself cruising in the Mediterranean as a crew member on a French coast guard boat, looking for evildoers. Periodically, your boat radios its position to headquarters in Marseille. You expect that communications will be intercepted. So, before you broadcast anything, you have to transform the actual position of the boat,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

( $x_1$  for Eastern longitude,  $x_2$  for Northern latitude), into an encoded position

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

You use the following code:

$$\begin{aligned} y_1 &= x_1 + 3x_2 \\ y_2 &= 2x_1 + 5x_2. \end{aligned}$$

For example, when the actual position of your boat is  $5^\circ$  E,  $42^\circ$  N, or

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 42 \end{bmatrix},$$

your encoded position will be

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 5 + 3 \cdot 42 \\ 2 \cdot 5 + 5 \cdot 42 \end{bmatrix} = \begin{bmatrix} 131 \\ 220 \end{bmatrix}.$$

See Figure 1.

The coding transformation can be represented as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}},$$

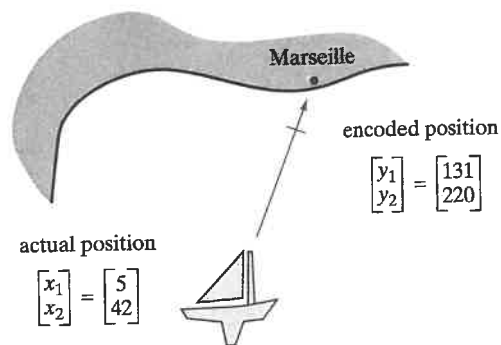


Figure 1

or, more succinctly, as

$$\vec{y} = A\vec{x}.$$

The matrix  $A$  is called the (*coefficient*) *matrix* of the transformation. A transformation of the form

$$\vec{y} = A\vec{x}$$

is called a *linear transformation*. We will discuss this important concept detail later in this section and throughout this chapter.

As the ship reaches a new position, the sailor on duty at headquarters receives the encoded message

$$\vec{b} = \begin{bmatrix} 133 \\ 223 \end{bmatrix}.$$

He must determine the actual position of the boat. He will have to solve system

$$A\vec{x} = \vec{b},$$

or, more explicitly,

$$\begin{cases} x_1 + 3x_2 = 133 \\ 2x_1 + 5x_2 = 223 \end{cases}.$$

Here is his solution. Is it correct?

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 43 \end{bmatrix}$$

As the boat travels on and dozens of positions are radioed in, the sailor gets a little tired of solving all those linear systems, and he thinks there must be a formula to simplify the task. He wants to solve the system

$$\begin{cases} x_1 + 3x_2 = y_1 \\ 2x_1 + 5x_2 = y_2 \end{cases}$$

when  $y_1$  and  $y_2$  are arbitrary constants, rather than particular numerical values. He is looking for the *decoding transformation*

$$\vec{y} \rightarrow \vec{x},$$

which is the *inverse*<sup>1</sup> of the coding transformation

$$\vec{x} \rightarrow \vec{y}.$$

The method of finding this solution is nothing new. We apply elimination as we have for a linear system with known values  $y_1$  and  $y_2$ :

$$\begin{array}{l} \left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ 2x_1 + 5x_2 = y_2 \end{array} \right| \xrightarrow{-2 \text{ (I)}} \left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ -x_2 = -2y_1 + y_2 \end{array} \right| \xrightarrow{\div(-1)} \\ \left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ x_2 = 2y_1 - y_2 \end{array} \right| \xrightarrow{-3 \text{ (II)}} \left| \begin{array}{l} x_1 = -5y_1 + 3y_2 \\ x_2 = 2y_1 - y_2 \end{array} \right| \end{array}$$

The formula for the decoding transformation is

$$\begin{aligned} x_1 &= -5y_1 + 3y_2, \\ x_2 &= 2y_1 - y_2, \end{aligned}$$

or

$$\vec{x} = B\vec{y}, \quad \text{where } B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

Note that the decoding transformation is linear and that its coefficient matrix is

$$B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

The relationship between the two matrices  $A$  and  $B$  is shown in Figure 2.

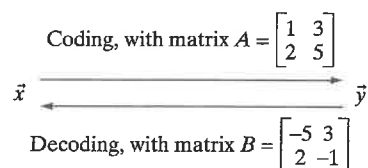


Figure 2

Since the decoding transformation  $\vec{x} = B\vec{y}$  is the inverse of the coding transformation  $\vec{y} = A\vec{x}$ , we say that the matrix  $B$  is the *inverse* of the matrix  $A$ . We can write this as  $B = A^{-1}$ .

Not all linear transformations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

are invertible. Suppose some ignorant officer chooses the code

$$\begin{aligned} y_1 &= x_1 + 2x_2 \\ y_2 &= 2x_1 + 4x_2 \end{aligned} \quad \text{with matrix } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

for the French coast guard boats. When the sailor in Marseille has to decode a position, for example,

$$\vec{b} = \begin{bmatrix} 89 \\ 178 \end{bmatrix},$$

<sup>1</sup>We will discuss the concept of the inverse of a transformation more systematically in Section 2.4.

he will be chagrined to discover that the system

$$\begin{cases} x_1 + 2x_2 = 89 \\ 2x_1 + 4x_2 = 178 \end{cases}$$

has infinitely many solutions, namely,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 89 - 2t \\ t \end{bmatrix},$$

where  $t$  is an arbitrary number.

Because this system does not have a unique solution, it is impossible to determine the actual position from the encoded position: The coding transformation and the decoding matrix  $A$  are *noninvertible*. This code is useless!

Now let us discuss the important concept of *linear transformations* in detail. Since linear transformations are a special class of functions, it may be helpful to review the concept of a *function* first.

Consider two sets  $X$  and  $Y$ . A function  $T$  from  $X$  to  $Y$  is a rule that associates with each element  $x$  of  $X$  a unique element  $y$  of  $Y$ . The set  $X$  is called the *domain* of the function, and  $Y$  is its *target space*. We will sometimes refer to  $x$  as the *input* of the function and to  $y$  as its *output*. Figure 3 shows an example where domain  $X$  and target space  $Y$  are finite.

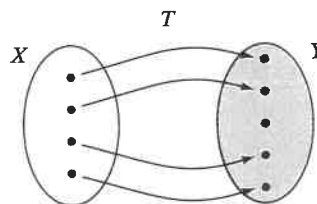


Figure 3 Domain  $X$  and target space  $Y$  of a function  $T$ .

In precalculus and calculus, you studied functions whose input and output are real numbers (i.e., whose domain and target space are the real numbers  $\mathbb{R}$  or  $\mathbb{C}$ ); for example,

$$y = x^2, \quad f(x) = e^x, \quad g(t) = \frac{t^2 - 2}{t - 1}.$$

In multivariable calculus, you may have encountered functions whose input and output were vectors.

#### EXAMPLE 1

$$y = x_1^2 + x_2^2 + x_3^2$$

This formula defines a function from the vector space  $\mathbb{R}^3$  to  $\mathbb{R}$ . The input is the vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ , and the output is the scalar  $y$ .

#### EXAMPLE 2

$$\vec{r} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

This formula defines a function from  $\mathbb{R}$  to the vector space  $\mathbb{R}^3$ , with input  $t$  and output  $\vec{r}$ .



**EXAMPLE 4** The coefficient matrix of the *identity transformation*

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ &\vdots \\ y_n &= x_n \end{aligned}$$

(a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  whose output equals its input) is the matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

All entries on the main diagonal are 1, and all other entries are 0. This is called the *identity matrix* and is denoted by  $I_n$ :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and so on.}$$

We have already seen the identity matrix in other contexts. For example, we have shown that a linear system  $A\vec{x} = \vec{b}$  of  $n$  equations with  $n$  unknowns has a unique solution if and only if  $\text{ref}(A) = I_n$ . See Theorem 1.3.4.

**EXAMPLE 5** Consider the letter L (for Linear?) in Figure 4, made up of the vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ . Show the effect of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

on this letter, and describe the transformation in words.

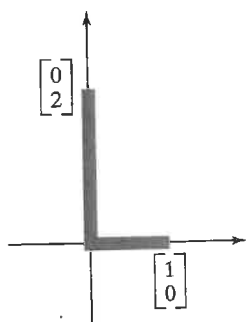


Figure 4

**Solution**

We have

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

as shown in Figure 5.

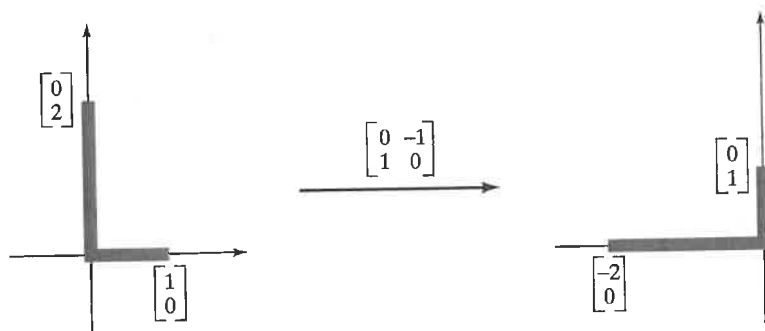


Figure 5

The  $L$  is rotated through an angle of  $90^\circ$  in the counterclockwise direction.

Let's examine the effect of transformation  $T$  on an arbitrary vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ :

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

We observe that the vectors  $\vec{x}$  and  $T(\vec{x})$  have the same length,

$$\sqrt{x_1^2 + x_2^2} = \sqrt{(-x_2)^2 + x_1^2},$$

and that they are perpendicular to one another, since the dot product equals zero. See Definition A.8 in the Appendix:

$$\vec{x} \cdot T(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = -x_1x_2 + x_2x_1 = 0.$$

Paying attention to the signs of the components, we see that if  $\vec{x}$  is in the first quadrant (meaning that  $x_1$  and  $x_2$  are both positive), then  $T(\vec{x}) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$  is in the second quadrant. See Figure 6.

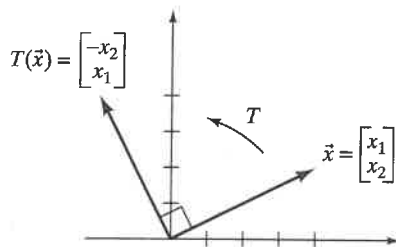


Figure 6

We can conclude that  $T(\vec{x})$  is obtained by rotating vector  $\vec{x}$  through an angle of  $90^\circ$  in the counterclockwise direction, as in the special cases  $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\vec{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$  considered earlier. (Check that the rotation is indeed counterclockwise when  $\vec{x}$  is in the second, third, or fourth quadrant.)

**EXAMPLE 6** Consider the linear transformation  $T(\vec{x}) = A\vec{x}$ , with

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where for simplicity we write  $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  instead of  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$ .

Solution

A straightforward computation shows that

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

and

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Note that  $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is the first column of the matrix  $A$  and that  $T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is column.

We can generalize this observation:

### Theorem 2.1.2

**The columns of the matrix of a linear transformation**

Consider a linear transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . Then, the matrix of

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}, \quad \text{where } \vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

To justify this result, write

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}.$$

Then

$$T(\vec{e}_i) = A\vec{e}_i = \begin{bmatrix} | & | & & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_i & \cdots & \vec{v}_m \\ | & | & & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i,$$

by Theorem 1.3.8.

The vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$  in the vector space  $\mathbb{R}^m$  are sometimes referred to as the *standard vectors* in  $\mathbb{R}^m$ . The standard vectors  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  in  $\mathbb{R}^3$  are often denoted by  $\vec{i}, \vec{j}, \vec{k}$ .



**EXAMPLE 7** Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .

- a. What is the relationship among  $T(\vec{v})$ ,  $T(\vec{w})$ , and  $T(\vec{v} + \vec{w})$ , where  $\vec{v}$  and  $\vec{w}$  are vectors in  $\mathbb{R}^m$ ?
- b. What is the relationship between  $T(\vec{v})$  and  $T(k\vec{v})$ , where  $\vec{v}$  is a vector in  $\mathbb{R}^m$  and  $k$  is a scalar?

**Solution**

- a. Applying Theorem 1.3.10, we find that

$$T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w}).$$

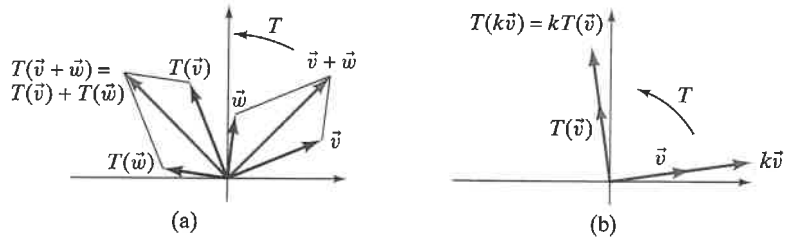
In words, the transform of the sum of two vectors equals the sum of the transforms.

- b. Again, apply Theorem 1.3.10:

$$T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v}).$$

In words, the transform of a scalar multiple of a vector is the scalar multiple of the transform. ■

Figure 7 illustrates these two properties in the case of the linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that rotates a vector through an angle of  $90^\circ$  in the counterclockwise direction. Compare this with Example 5.



**Figure 7** (a) Illustrating the property  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ .  
 (b) Illustrating the property  $T(k\vec{v}) = kT(\vec{v})$ .

In Example 7, we saw that a linear transformation satisfies the two equations  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$  and  $T(k\vec{v}) = kT(\vec{v})$ . Now we will show that the converse is true as well: Any transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  that satisfies these two equations is a linear transformation.

**Theorem 2.1.3**

**Linear transformations**

A transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  is linear if (and only if)

- a.  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ , for all vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^m$ , and
- b.  $T(k\vec{v}) = kT(\vec{v})$ , for all vectors  $\vec{v}$  in  $\mathbb{R}^m$  and all scalars  $k$ .

**Proof** In Example 7, we saw that a linear transformation satisfies the equations in (a) and (b). To prove the converse, consider a transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  that satisfies equations (a) and (b). We must show that there exists a matrix  $A$  such that

ird ■

to as noted

$T(\vec{x}) = A\vec{x}$ , for all  $\vec{x}$  in the vector space  $\mathbb{R}^m$ . Let  $\vec{e}_1, \dots, \vec{e}_m$  be the standard introduced in Theorem 2.1.2.

$$\begin{aligned} T(\vec{x}) &= T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_m\vec{e}_m) \quad (\text{by property a}) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_mT(\vec{e}_m) \quad (\text{by property b}) \\ &= \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = A\vec{x} \end{aligned}$$

Here is an example illustrating Theorem 2.1.3.

**EXAMPLE 8** Consider a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  such that  $T(\vec{v}_1) = T(\vec{v}_2) = 2\vec{v}_2$ , for the vectors  $\vec{v}_1$  and  $\vec{v}_2$  sketched in Figure 8. On the sketch  $T(\vec{x})$ , for the given vector  $\vec{x}$ . Explain your solution.

**Solution**

Using a parallelogram, we can represent  $\vec{x}$  as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$  as shown in Figure 9:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2.$$

By Theorem 2.1.3,

$$T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = \frac{1}{2}c_1\vec{v}_1 + 2c_2\vec{v}_2$$

The vector  $c_1\vec{v}_1$  is cut in half, and the vector  $c_2\vec{v}_2$  is doubled, as shown in Figure 10.

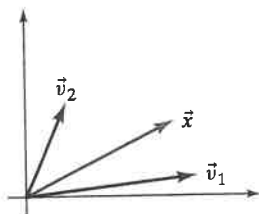


Figure 8

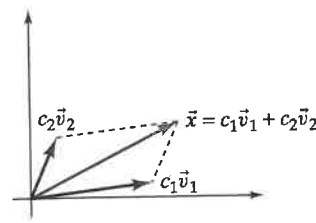


Figure 9

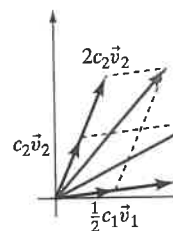
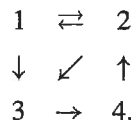


Figure 10

Imagine that vector  $\vec{x}$  is drawn on a rubber sheet. Transformation  $T$  stretches this sheet by a factor of 2 in the  $\vec{v}_2$ -direction and contracts it by a factor of  $\frac{1}{2}$  in the  $\vec{v}_1$ -direction. (We prefer “contracts by a factor of 2” to the awkward “expands by a factor of  $\frac{1}{2}$ .”)

We will conclude this section with one more example of a linear transformation from computer science.

**EXAMPLE 9** Let's develop a simple model of how people might surf the World Wide Web, by following links. To keep things manageable, we consider a "mini-Web" with only four pages, labeled 1, 2, 3, 4, linked as shown in the diagram below:



Let  $x_1, x_2, x_3,$  and  $x_4$  be the proportions of the surfers who find themselves on each of the four pages initially; we can collect this information in the *distribution vector*

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \text{ For example, the initial distribution might be } \vec{x} = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{bmatrix},$$

meaning that 40% of the surfers are on page 1, and so forth. The components of a distribution vector add up to 1, or 100%, of course.

At a predetermined time, at the blow of a whistle, each surfer will randomly follow one of the links: If several links are available out of a web page, then an equal proportion of the surfers will follow each of them. For example, the proportion of surfers taking each of the two links out of page 1 (to pages 2 and 3) will be  $\frac{x_1}{2}$ .

Let the vector  $\vec{y}$ , with components  $y_1, y_2, y_3, y_4$ , represent the distribution of the surfers after the transition. According to the rules stated above, we will have

$$\begin{aligned} y_1 &= \frac{1}{2}x_2 \\ y_2 &= \frac{1}{2}x_1 + x_4 \\ y_3 &= \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ y_4 &= x_3 \end{aligned}$$

or  $\vec{y} = A\vec{x}$  in vector form, where

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

showing that  $\vec{y} = T(\vec{x})$  is a linear transformation. Matrix  $A$  is referred to as the *transition matrix* of this transformation.

For example, the equation  $y_2 = \frac{1}{2}x_1 + x_4$  reflects the fact that half of the surfers from page 1 and all of the surfers from page 4 will follow the links to page 2.

Alternatively, we can construct the matrix  $A$  column by column. The  $j$ th column of  $A$  tells us where the surfers go from page  $j$ . For example, the second column of  $A$  is

$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix},$$

indicating that half of the surfers from page 2 take the link to page 1, and the other half go to page 3. Thus, the entries in each column of  $A$  must add up to 1.

Let's discuss the structure of matrix  $A$  more formally. Let  $c_j$  be the number of links going out of page  $j$ . For example, we have  $c_2 = 2$  and  $c_3 = 1$  for our

mini-Web. Then the proportion of the surfers taking a link from page  $j$  to page  $i$  will be  $\frac{x_j}{c_j}$ , since the initial population  $x_j$  of page  $j$  gets distributed equally among the  $c_j$  outgoing links. We see that the  $ij$ th entry of  $A$  is  $\frac{1}{c_j}$  if there is a link from page  $j$  to page  $i$ ; the other entries are 0. Thus, we have

$$a_{ij} = \begin{cases} 1/c_j & \text{if there is a link } j \rightarrow i \\ 0 & \text{otherwise.} \end{cases}$$

We might wonder whether this system has any *equilibrium distributions*, that is, distributions  $\vec{x}$  such that  $A\vec{x} = \vec{x}$ . The distribution after the transition is exactly the same as before. To find out, we have to solve the linear system

$$\begin{cases} \frac{1}{2}x_2 = x_1 \\ \frac{1}{2}x_1 + x_4 = x_2 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 = x_3 \\ x_3 = x_4 \end{cases} \quad \text{or} \quad \begin{cases} -x_1 + \frac{1}{2}x_2 = 0 \\ \frac{1}{2}x_1 - x_2 + x_4 = 0 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - x_3 = 0 \\ x_3 - x_4 = 0 \end{cases}$$

with augmented matrix  $M = \begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$  and reduced

$$\begin{bmatrix} 1 & 0 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & -4/3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{The solutions are } x_1 = \frac{2t}{3}, x_2 = \frac{4t}{3}, x_3 = t$$

where  $t$  is an arbitrary real number. Since we are looking for a distribution where  $x_1 + x_2 + x_3 + x_4 = 4t = 1$ , so  $t = \frac{1}{4}$ . The equilibrium distribution is

$$\vec{x}_{equ} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix} \approx \begin{bmatrix} 16.7\% \\ 33.3\% \\ 25\% \\ 25\% \end{bmatrix}$$

In this context, an interesting question arises: If we iterate our transition matrix, will the surfers move to a new page over and over again, following links a will the system eventually approach this equilibrium state  $\vec{x}_{equ}$ , regardless of the initial distribution? Perhaps surprisingly, the answer is affirmative for the system considered in this example, as well as for many others: *The equilibrium distribution represents the distribution of the surfers in the long run, for any initial distribution.* We will further discuss these important issues in Theorem 2.3.11 and the next chapter.

In 1998, Sergey Brin and Lawrence Page, then at Stanford University, published a landmark paper, "The Anatomy of a Large-Scale Hypertextual Search Engine," where they present a prototype of the search engine Google. The key feature of the search engine is a "quality ranking" of each web page, which measures the page's popularity. They call this ranking PageRank, after the second author of the paper.

The basic idea is to define the popularity of a web page as the likelihood that a random surfer finds themselves on that web page in the long run. But this is exactly the corresponding component of the equilibrium vector  $\vec{x}_{equ}$  we found above. For example, page 2 is the most popular, with a PageRank of  $1/3$ , while page 1 is half as popular, with a PageRank of  $1/6$ .

Actually, the model developed by Brin and Page in their 1998 paper is a bit more complicated, involving a “jumping rate” as well. They assume that surfers will not always follow links, but sometimes randomly jump to a new page, even if there is no link available to that page. The “jumping rate,” often set at 0.15 or 0.2, represents the proportion of transitions where a surfer jumps rather than following a link. See Exercise 53. In our example, we set the jumping rate to be 0 in order to simplify the computation. The resulting popularity ranking (with jumping rate 0) is sometimes referred to as the *naïve PageRank*. ■

Example 9 motivates the following definitions:

**Definition 2.1.4 Distribution vectors and transition matrices**

A vector  $\vec{x}$  in  $\mathbb{R}^n$  is said to be a *distribution vector* if its components add up to 1 and all the components are positive or zero.

A square matrix  $A$  is said to be a *transition matrix* (or stochastic matrix) if all its columns are distribution vectors. This means that all the entries of a transition matrix are positive or zero, and the entries in each column add up to 1.

If  $A$  is a transition matrix and  $\vec{x}$  is a distribution vector, then  $A\vec{x}$  will be a distribution vector as well. See Exercise 49.

In Example 9, the vector  $\vec{x}_{equ}$  is a distribution vector and the matrix  $A$  is a transition matrix. Simple examples of transition matrices are  $\begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0.2 & 0 \\ 0.8 & 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ , while  $\begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 1.1 \end{bmatrix}$  fails to be a transition matrix since one of its entries is negative.

**EXERCISES 2.1**

**GOAL** Use the concept of a linear transformation in terms of the formula  $\vec{y} = A\vec{x}$ , and interpret simple linear transformations geometrically. Find the inverse of a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  (if it exists). Find the matrix of a linear transformation column by column.

Consider the transformations from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined in Exercises 1 through 3. Which of these transformations are linear?

- 1.  $y_1 = 2x_2$       2.  $y_1 = 2x_2$       3.  $y_1 = x_2 - x_3$   
 $y_2 = x_2 + 2$        $y_2 = 3x_3$        $y_2 = x_1x_3$   
 $y_3 = 2x_2$        $y_3 = x_1$        $y_3 = x_1 - x_2$

4. Find the matrix of the linear transformation

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3. \end{aligned}$$

5. Consider the linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^2$  with

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix},$$

$$\text{and } T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix}.$$

Find the matrix  $A$  of  $T$ .

6. Consider the transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^3$  given by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Is this transformation linear? If so, find its matrix.

7. Suppose  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  are arbitrary vectors in  $\mathbb{R}^n$ . Consider the transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m.$$

Is this transformation linear? If so, find its matrix  $A$  in terms of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ .

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8. Find the inverse of the linear transformation

$$\begin{aligned} y_1 &= x_1 + 7x_2 \\ y_2 &= 3x_1 + 20x_2. \end{aligned}$$

In Exercises 9 through 12, decide whether the given matrix is invertible. Find the inverse if it exists. In Exercise 12, the constant  $k$  is arbitrary.

9.  $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$       10.  $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

11.  $\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$       12.  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

13. Prove the following facts:

- a. The  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if  $ad - bc \neq 0$ . *Hint:* Consider the cases  $a \neq 0$  and  $a = 0$  separately.

- b. If

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

[The formula in part (b) is worth memorizing.]

14. a. For which values of the constant  $k$  is the matrix  $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$  invertible?

- b. For which values of the constant  $k$  are all entries of  $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$  integers?

See Exercise 13.

15. For which values of the constants  $a$  and  $b$  is the matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

invertible? What is the inverse in this case? See Exercise 13.

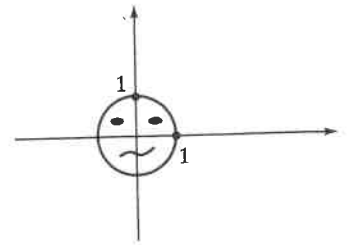
Give a geometric interpretation of the linear transformations defined by the matrices in Exercises 16 through 23. Show the effect of these transformations on the letter L considered in Example 5. In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically. See Exercise 13.

16.  $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$       17.  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$       18.  $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$

19.  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$       20.  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$       21.  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

22.  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$       23.  $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Consider the circular face in the accompanying figure each of the matrices  $A$  in Exercises 24 through a sketch showing the effect of the linear transformation  $T(\vec{x}) = A\vec{x}$  on this face.



24.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$       25.  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$       26.  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

27.  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$       28.  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$       29.  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

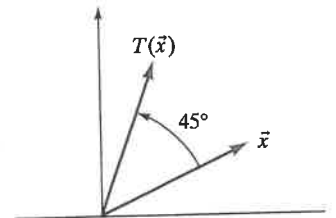
30.  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

31. In Chapter 1, we mentioned that an old G shows the mirror image of Gauss's likeness. A transformation  $T$  can you apply to get the act back?



32. Find an  $n \times n$  matrix  $A$  such that  $A\vec{x} = \vec{x}$  in  $\mathbb{R}^n$ .

33. Consider the transformation  $T$  from  $\mathbb{R}^2$  rotates any vector  $\vec{x}$  through an angle of  $45^\circ$  counterclockwise direction, as shown in the follo



You are told that  $T$  is a linear transformation (to be shown in the next section.) Find the ma

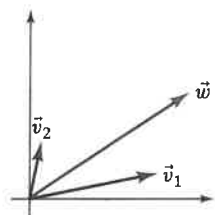
34. Consider the transformation  $T$  from  $\mathbb{R}^2$  rotates any vector  $\vec{x}$  through a given angle  $\theta$  counterclockwise direction. Compare this with You are told that  $T$  is linear. Find the m terms of  $\theta$ .

35. In the example about the French coast guard in this section, suppose you are a spy watching the boat and listening in on the radio messages from the boat. You collect the following data:

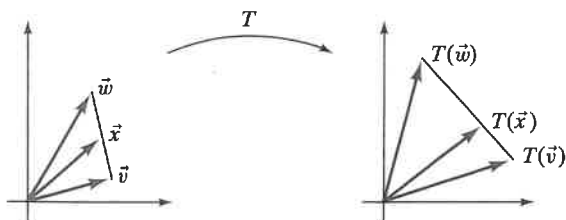
- When the actual position is  $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$ , they radio  $\begin{bmatrix} 89 \\ 52 \end{bmatrix}$ .
- When the actual position is  $\begin{bmatrix} 6 \\ 41 \end{bmatrix}$ , they radio  $\begin{bmatrix} 88 \\ 53 \end{bmatrix}$ .

Can you crack their code (i.e., find the coding matrix), assuming that the code is linear?

36. Let  $T$  be a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Let  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{w}$  be three vectors in  $\mathbb{R}^2$ , as shown below. We are told that  $T(\vec{v}_1) = \vec{v}_1$  and  $T(\vec{v}_2) = 3\vec{v}_2$ . On the same axes, sketch  $T(\vec{w})$ .



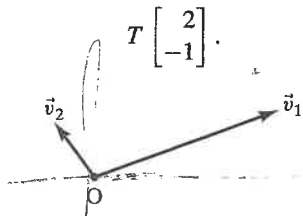
37. Consider a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Suppose that  $\vec{v}$  and  $\vec{w}$  are two arbitrary vectors in  $\mathbb{R}^2$  and that  $\vec{x}$  is a third vector whose endpoint is on the line segment connecting the endpoints of  $\vec{v}$  and  $\vec{w}$ . Is the endpoint of the vector  $T(\vec{x})$  necessarily on the line segment connecting the endpoints of  $T(\vec{v})$  and  $T(\vec{w})$ ? Justify your answer.



Hint: We can write  $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$ , for some scalar  $k$  between 0 and 1.

We can summarize this exercise by saying that a linear transformation maps a line onto a line.

38. The two column vectors  $\vec{v}_1$  and  $\vec{v}_2$  of a  $2 \times 2$  matrix  $A$  are shown in the accompanying sketch. Consider the linear transformation  $T(\vec{x}) = A\vec{x}$ , from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Sketch the vector



39. Show that if  $T$  is a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , then

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_m T(\vec{e}_m),$$

where  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$  are the standard vectors in  $\mathbb{R}^m$ .

40. Describe all linear transformations from  $\mathbb{R}$  ( $= \mathbb{R}^1$ ) to  $\mathbb{R}$ . What do their graphs look like?

41. Describe all linear transformations from  $\mathbb{R}^2$  to  $\mathbb{R}$  ( $= \mathbb{R}^1$ ). What do their graphs look like?

42. When you represent a three-dimensional object graphically in the plane (on paper, the blackboard, or a computer screen), you have to transform spatial coordinates,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

into plane coordinates,  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ . The simplest choice is a linear transformation, for example, the one given by the matrix

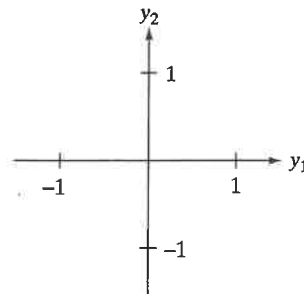
$$\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Include the images of the  $x_1, x_2,$  and  $x_3$  axes in your sketch:



b. Represent the image of the point  $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$  in your figure in part (a).

(part c on next page)

c. Find all the points

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$

that are transformed to  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . Explain.

43. a. Consider the vector  $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ . Is the transformation

$T(\vec{x}) = \vec{v} \cdot \vec{x}$  (the dot product) from  $\mathbb{R}^3$  to  $\mathbb{R}$  linear? If so, find the matrix of  $T$ .

b. Consider an arbitrary vector  $\vec{v}$  in  $\mathbb{R}^3$ . Is the transformation  $T(\vec{x}) = \vec{v} \cdot \vec{x}$  linear? If so, find the matrix of  $T$  (in terms of the components of  $\vec{v}$ ).

c. Conversely, consider a linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}$ . Show that there exists a vector  $\vec{v}$  in  $\mathbb{R}^3$  such that  $T(\vec{x}) = \vec{v} \cdot \vec{x}$ , for all  $\vec{x}$  in  $\mathbb{R}^3$ .

44. The cross product of two vectors in  $\mathbb{R}^3$  is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

See Definition A.9 and Theorem A.11 in the Appendix. Consider an arbitrary vector  $\vec{v}$  in  $\mathbb{R}^3$ . Is the transformation  $T(\vec{x}) = \vec{v} \times \vec{x}$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  linear? If so, find its matrix in terms of the components of the vector  $\vec{v}$ .

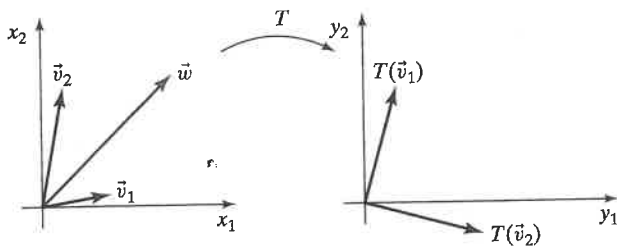
45. Consider two linear transformations  $\vec{y} = T(\vec{x})$  and  $\vec{z} = L(\vec{y})$ , where  $T$  goes from  $\mathbb{R}^m$  to  $\mathbb{R}^p$  and  $L$  goes from  $\mathbb{R}^p$  to  $\mathbb{R}^n$ . Is the transformation  $\vec{z} = L(T(\vec{x}))$  linear as well? [The transformation  $\vec{z} = L(T(\vec{x}))$  is called the *composite* of  $T$  and  $L$ .]

46. Let

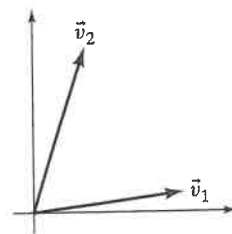
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Find the matrix of the linear transformation  $T(\vec{x}) = B(A\vec{x})$ . See Exercise 45. *Hint*: Find  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$ .

47. Let  $T$  be a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Three vectors  $\vec{v}_1, \vec{v}_2, \vec{w}$  in  $\mathbb{R}^2$  and the vectors  $T(\vec{v}_1), T(\vec{v}_2)$  are shown in the accompanying figure. Sketch  $T(\vec{w})$ . Explain your answer.



48. Consider two linear transformations  $T$  and  $L$  to  $\mathbb{R}^2$ . We are told that  $T(\vec{v}_1) = L(\vec{v}_1)$  and  $L(\vec{v}_2)$  for the vectors  $\vec{v}_1$  and  $\vec{v}_2$  sketched below that  $T(\vec{x}) = L(\vec{x})$ , for all vectors  $\vec{x}$  in  $\mathbb{R}^2$ .



49. Prove that if  $A$  is a transition matrix and  $\vec{x}$  is a distribution vector, then  $A\vec{x}$  is a distribution vector.

For each of the mini-Webs in Exercises 50 through 53,

- find the transition matrix  $A$  as defined in the diagram;
- find the equilibrium distribution, and
- find the web page(s) with the highest (net) Rank.

Feel free to use technology throughout.

50.  $1 \rightleftharpoons 2$   
 $\downarrow \nearrow \uparrow$   
 $3 \rightarrow 4$

51.  $1 \rightarrow 2$   
 $\uparrow \nearrow \downarrow$   
 $3 \rightleftharpoons 4$

52.  $1 \rightleftharpoons 2$   
 $\downarrow \nearrow$   
 $3$

53. As in Example 9, consider the mini-Web structure shown in the diagram below:

$1 \rightleftharpoons 2$   
 $\downarrow \nearrow \uparrow$   
 $3 \rightarrow 4.$

Again, let vector  $\vec{x}$  represent the distribution among the various pages at a given time. For the 1998 paper by Brin and Page (see Example 9), consider a more sophisticated model of transition. Suppose that 80% of the surfers on a web page will randomly follow a link, while the remaining 20% will randomly "jump" to a web page (even the current page). For example, of the surfers on page 2, 40% will follow the link to page 1, 40% will follow the link to page 3, and 5% will randomly jump to any of the four webpages. If we add up the probabilities for all the link-followers, we see that 45% will go to page 1, 45% will go to page 3, 5% will stay on page 2. Thus, the second column of the transition matrix  $B$  will be



$\mathbb{R}^2$   
=  
JW

$$\begin{bmatrix} 0.4 + 0.05 & \\ & 0.05 \\ 0.4 + 0.05 & \\ & 0.05 \end{bmatrix} = \begin{bmatrix} 0.45 & \\ & 0.05 \\ 0.45 & \\ & 0.05 \end{bmatrix}$$

$\begin{matrix} \uparrow & \swarrow \\ \text{link} & \text{jumpers} \\ \text{followers} \end{matrix}$

Let  $\vec{y}$  be the distribution vector after the transition; we have  $y_2 = 0.45x_1 + 0.05x_2 + 0.05x_3 + 0.85x_4$ , for example.

- Find the transition matrix  $B$  such that  $\vec{y} = B\vec{x}$ .
- Explain why  $B = \frac{0.2}{N}E + 0.8A = 0.05E + 0.8A$ , where  $A$  is the matrix we found in Example 9,  $E$  is the matrix whose entries are all 1, and  $N$  is the total number of web pages.
- Find the equilibrium distribution  $\vec{x}_{equ}$ , such that  $B\vec{x}_{equ} = \vec{x}_{equ}$ . Feel free to use technology. (The components of this equilibrium solution represent the PageRank of the various web pages, as defined in the 1998 paper by Brin and Page.)

For each of the mini-Webs in Exercises 54 through 56,

- find the transition matrix  $B$  as defined in Exercise 53 (with a jumping rate of 20%),
- find the equilibrium distribution for the transition matrix  $B$ , and
- find the web page(s) with the highest PageRank (with a jumping rate of 20%).

Feel free to use technology throughout.

54.  $1 \rightleftharpoons 2$   
 $\downarrow \nearrow \uparrow$   
 $3 \rightarrow 4$

55.  $1 \rightarrow 2$   
 $\updownarrow \nearrow \updownarrow$   
 $3 \rightleftharpoons 4$

56.  $1 \rightleftharpoons 2$   
 $\downarrow \nearrow$   
 $3$

57. Some parking meters in downtown Geneva, Switzerland, accept 2 Franc and 5 Franc coins.

- A parking officer collects 51 coins worth 144 Francs. How many coins are there of each kind?
- Find the matrix  $A$  that transforms the vector

$$\begin{bmatrix} \text{number of 2 Franc coins} \\ \text{number of 5 Franc coins} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix}$$

- Is the matrix  $A$  in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

58. A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.

- King Hiero of Syracuse orders a crown from this goldsmith, with a total mass of 5 kilograms (or 5,000 grams), with the stipulation that the platinum alloy must make up at least 90% of the mass. The goldsmith delivers a beautiful piece, but the king's friend Archimedes has doubts about its purity. While taking a bath, he comes up with a method to check the composition of the crown (famously shouting "Eureka!" in the process, and running to the king's palace naked). Submerging the crown in water, he finds its volume to be 370 cubic centimeters. How much of each alloy went into this piece (by mass)? Is this goldsmith a crook?
- Find the matrix  $A$  that transforms the vector

$$\begin{bmatrix} \text{mass of platinum alloy} \\ \text{mass of silver alloy} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix},$$

for any piece of jewelry this goldsmith makes.

- Is the matrix  $A$  in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

59. The conversion formula  $C = \frac{5}{9}(F - 32)$  from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.

- Find the  $2 \times 2$  matrix  $A$  that transforms the vector  $\begin{bmatrix} F \\ 1 \end{bmatrix}$  into the vector  $\begin{bmatrix} C \\ 1 \end{bmatrix}$ . (The second row of  $A$  will be  $[0 \ 1]$ .)
- Is the matrix  $A$  in part (a) invertible? If so, find the inverse (use Exercise 13). Use the result to write a formula expressing  $F$  in terms of  $C$ .

60. In the financial pages of a newspaper, one can sometimes find a table (or matrix) listing the exchange rates between currencies. In this exercise we will consider a miniature version of such a table, involving only the Canadian dollar (C\$) and the South African Rand (ZAR). Consider the matrix

$$A = \begin{bmatrix} \text{C\$} & \text{ZAR} \\ 1 & 1/8 \\ 8 & 1 \end{bmatrix} \begin{matrix} \text{C\$} \\ \text{ZAR} \end{matrix}$$

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representing the fact that C\$1 is worth ZAR8 (as of September 2012).

- a. After a trip you have C\$100 and ZAR1,600 in your pocket. We represent these two values in the vector

$\vec{x} = \begin{bmatrix} 100 \\ 1,600 \end{bmatrix}$ . Compute  $A\vec{x}$ . What is the practical significance of the two components of the vector  $A\vec{x}$ ?

- b. Verify that matrix  $A$  fails to be invertible. For which vectors  $\vec{b}$  is the system  $A\vec{x} = \vec{b}$  consistent? What is the practical significance of your answer? If the system  $A\vec{x} = \vec{b}$  is consistent, how many solutions  $\vec{x}$  are there? Again, what is the practical significance of the answer?

61. Consider a larger currency exchange matrix (see Exercise 60), involving four of the world's leading currencies: Euro (€), U.S. dollar (\$), Chinese yuan (¥), and British pound (£).

$$A = \begin{array}{cccc|c} \text{€} & \$ & \text{¥} & \text{£} & \\ \hline * & 0.8 & * & * & \text{€} \\ * & * & * & * & \$ \\ * & * & * & 10 & \text{¥} \\ 0.8 & * & * & * & \text{£} \end{array}$$

The entry  $a_{ij}$  gives the value of one unit of the  $j$ th currency, expressed in terms of the  $i$ th currency. For example,  $a_{34} = 10$  means that £1 = ¥10 (as of August 2012). Find the exact values of the 13 missing entries of  $A$  (expressed as fractions).

62. Consider an arbitrary currency exchange matrix  $A$ . Exercises 60 and 61.

- a. What are the diagonal entries  $a_{ii}$  of  $A$ ?  
 b. What is the relationship between  $a_{ij}$  and  $a_{ji}$ ?  
 c. What is the relationship among  $a_{ik}$ ,  $a_{kj}$ , and  $a_{ij}$ ?  
 d. What is the rank of  $A$ ? What is the relationship between  $A$  and  $\text{rref}(A)$ ?

63. Solving a linear system  $A\vec{x} = \vec{b}$  by Gaussian elimination amounts to writing the vector of leading ones as a linear transformation of the vector of free variables. Consider the linear system

$$\begin{array}{rcl} x_1 - x_2 & + & 4x_5 = 0 \\ & x_3 & - x_5 = 0 \\ & & x_4 - 2x_5 = 0. \end{array}$$

Find the matrix  $B$  such that  $\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = B \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$

64. Consider the linear system

$$\begin{array}{rcl} x_1 + 2x_2 + x_3 + 7x_4 & = & 0 \\ x_1 + 2x_2 + 2x_3 + 11x_4 & = & 0 \\ x_1 + 2x_2 + 3x_3 + 15x_4 & = & 0 \\ x_1 + 2x_2 + 4x_3 + 19x_4 & = & 0. \end{array}$$

Find the matrix  $B$  such that  $\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = B \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$

Exercise 63.

## 2.2 Linear Transformations in Geometry

In Example 2.1.5 we saw that the matrix  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  represents a counterclockwise rotation through  $90^\circ$  in the coordinate plane. Many other  $2 \times 2$  matrices represent simple geometrical transformations as well; this section is dedicated to a study of some of those transformations.

**EXAMPLE 1** Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Show the effect of each of these matrices on our standard letter L,<sup>3</sup> as a result of each transformation in words.

<sup>3</sup>See Example 2.1.5. Recall that vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is the foot of our standard L, and  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  is its top.