1.3 On the Solutions of Linear Systems; Matrix Algebra

In this final section of Chapter 1, we will discuss two rather unrelated topics:

- First, we will examine how many solutions a system of linear equations can possibly have.
- Then, we will present some definitions and rules of matrix algebra.

The Number of Solutions of a Linear System

EXAMPLE 1 The reduced row-echelon forms of the augmented matrices of three systems are given. How many solutions are there in each case?

$$\textbf{a.}\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \textbf{b.}\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \textbf{c.}\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution

a. The third row represents the equation $0 = 1$, so that there are no solutions. We say that this system is *inconsistent*.

b. The given augmented matrix represents the system

$$\left| \begin{aligned} x_1 + 2x_2 \quad &= 1 \\ x_3 &= 2 \end{aligned} \right|, \qquad \text{or} \qquad \left| \begin{aligned} x_1 &= 1 - 2x_2 \\ x_3 &= 2 \end{aligned} \right|.$$

We can assign an arbitrary value, t, to the free variable x_2, so that the system has infinitely many solutions,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 2t \\ t \\ 2 \end{bmatrix}, \qquad \text{where } t \text{ is an arbitrary constant.}$$

c. Here there are no free variables, so that we have only one solution, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. ■

We can generalize our findings:[16]

Theorem 1.3.1

Number of solutions of a linear system

A system of equations is said to be *consistent* if there is at least one solution; it is *inconsistent* if there are no solutions.

A linear system is inconsistent if (and only if) the reduced row-echelon form of its augmented matrix contains the row $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$, representing the equation $0 = 1$.

If a linear system is consistent, then it has either

- *infinitely many solutions* (if there is at least one free variable), or
- *exactly one solution* (if all the variables are leading).

[16] Starting in this section, we will number the definitions we give and the theorems we derive. The nth theorem stated in Section $p.q$ is labeled as Theorem $p.q.n$.

Example 1 illustrates what the number of leading 1's in the echelon fo
us about the number of solutions of a linear system. This observation motiv
following definition:

Definition 1.3.2 **The rank of a matrix**[17]

The rank of a matrix A is the number of leading 1's in rref(A), denoted ra

EXAMPLE 2

$$\text{rank}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = 2, \text{ since rref}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = \begin{bmatrix}\textcircled{1} & 0 & -1\\0 & \textcircled{1} & 2\\0 & 0 & 0\end{bmatrix}$$

Note that we have defined the rank of a *matrix* rather than the rank of
system. When relating the concept of rank to a linear system, we must be
to specify whether we consider the coefficient matrix or the augmented m
the system.

EXAMPLE 3 Consider a system of n linear equations with m variables, which has a co
matrix A of size $n \times m$. Show that

a. The inequalities $\text{rank}(A) \le n$ and $\text{rank}(A) \le m$ hold.

b. If the system is inconsistent, then $\text{rank}(A) < n$.

c. If the system has exactly one solution, then $\text{rank}(A) = m$.

d. If the system has infinitely many solutions, then $\text{rank}(A) < m$.

Solution

a. By definition of the reduced row-echelon form, there is at most one
1 in each of the n rows and in each of the m columns of rref(A).

b. If the system is inconsistent, then the rref of the augmented matrix
tain a row of the form [0 0 ... 0 1], so that rref(A) will contain
zeros. Since there is no leading 1 in that row, we find that rank(A)
claimed.

c. For parts c and d, it is worth noting that

$$\begin{pmatrix}\text{number of}\\\text{free variables}\end{pmatrix} = \begin{pmatrix}\text{total number}\\\text{of variables}\end{pmatrix} - \begin{pmatrix}\text{number of}\\\text{leading variables}\end{pmatrix} = m -$$

If the system has exactly one solution, then there are no free varia
Theorem 1.3.1), so that $m - \text{rank}(A) = 0$ and $\text{rank}(A) = m$ as clai

d. If the system has infinitely many solutions, then there is at least
variable, so that $m - \text{rank}(A) > 0$ and $\text{rank}(A) < m$, as claimed.

EXAMPLE 4 It is useful to think about the *contrapositives* of the statements in parts b t
of Example 3.[18]

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[17] This is a preliminary, rather technical definition. In Chapter 3, we will gain a better conce
understanding of the rank.

[18] The *contrapositive* of the statement "if p then q" is "if not-q then not-p." A statement an
contrapositive are logically equivalent. For example, the contrapositive of "If you live in N
City, then you live in the Unites States" is "If you don't live in the United States, then you
New York City." Here is a more convoluted example: On the service truck of a plumbing c
read, "If we can't fix it, then it ain't broken." The contrapositive of this claim is, "If it is br
we can fix it" (not quite as catchy!).

b. If $\text{rank}(A) = n$, then the system is consistent.

c. If $\text{rank}(A) < m$, then the system has no solution or infinitely many solutions.

d. If $\text{rank}(A) = m$, then the system has no solution or exactly one solution. ■

In Theorems 1.3.3 and 1.3.4, we will discuss two important special cases of Example 3.

Theorem 1.3.3

Number of equations vs. number of unknowns

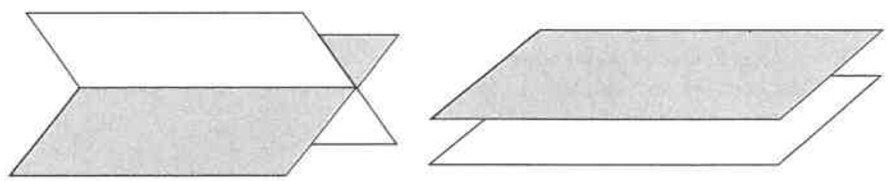
a. If a linear system has exactly one solution, then there must be at least as many equations as there are variables ($m \leq n$ with the notation from Example 3).

Equivalently, we can formulate the contrapositive:

b. A linear system with fewer equations than unknowns ($n < m$) has either no solutions or infinitely many solutions.

The proof of part (a) is based on parts (a) and (c) of Example 3: $m = \text{rank}(A) \leq n$, so that $m \leq n$ as claimed.

To illustrate part b of Theorem 1.3.3, consider two linear equations in three variables, with each equation defining a plane. Two different planes in space either intersect in a line or are parallel (see Figure 1), but they will never intersect at a point! This means that a system of two linear equations with three unknowns cannot have a unique solution.



(a) (b)

Figure 1 (a) Two planes intersect in a line. (b) Two parallel planes.

EXAMPLE 5

Consider a linear system of n equations with n variables. When does this system have exactly one solution? Give your answer in terms of the rank of the coefficient matrix A.

Solution

If the system has exactly one solution, then $\text{rank}(A) = m = n$ by Example 3c.

Conversely, if $\text{rank}(A) = n$, then there will be a leading 1 in each row and in each column, and these leading 1's will be lined up along the diagonal:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

This system will have exactly one solution. ■

Theorem 1.3.4 **Systems of n equations in n variables**

A linear system of n equations in n variables has a unique solution if (an if) the rank of its coefficient matrix A is n. In this case,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

the $n \times n$ matrix with 1's along the diagonal and 0's everywhere else.

Matrix Algebra

We will now introduce some basic definitions and rules of matrix algebra. (sentation will be somewhat lacking in motivation at first, but it will be gooc these tools available when we need them in Chapter 2.

Sums and scalar multiples of matrices are defined entry by entry, as for See Definition A.1 in the Appendix.

Definition 1.3.5 **Sums of matrices**

The sum of two matrices of the same size is defined entry by entry:

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1m}+b \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \dots & a_{nm}+b \end{bmatrix}$$

Scalar multiples of matrices

The product of a scalar with a matrix is defined entry by entry:

$$k\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \dots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \dots & ka_{nm} \end{bmatrix}.$$

EXAMPLE 6 $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 4 \\ 9 & 8 & 5 \end{bmatrix}$$

EXAMPLE 7 $$3\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}$$

The definition of the product of matrices is less straightforward; we the general definition later in Section 2.3.

Because vectors are special matrices (with only one row or only one it makes sense to start with a discussion of products of vectors. The reade familiar with the dot product of vectors.

Definition 1.3.6 **Dot product of vectors**

Consider two vectors $\vec{v}$ and $\vec{w}$ with components $v_1, \dots, v_n$ and $w_1, \dots, w_n$, respectively. Here $\vec{v}$ and $\vec{w}$ may be column or row vectors, and the two vectors need not be of the same type. The dot product of $\vec{v}$ and $\vec{w}$ is defined to be the scalar

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n.$$

Note that our definition of the dot product isn't row-column-sensitive. The dot product does not distinguish between row and column vectors.

EXAMPLE 8 $$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 = 11$$ ■

Now we are ready to define the product $A\vec{x}$, where A is a matrix and $\vec{x}$ is a vector, in terms of the dot product.

Definition 1.3.7 **The product $A\vec{x}$**

If A is an $n \times m$ matrix with row vectors $\vec{w}_1, \dots, \vec{w}_n$, and $\vec{x}$ is a vector in $\mathbb{R}^m$, then

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}.$$

In words, the ith component of $A\vec{x}$ is the dot product of the ith row of A with $\vec{x}$.

Note that $A\vec{x}$ is a column vector with n components, that is, a vector in $\mathbb{R}^n$.

EXAMPLE 9 $$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 + (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$ ■

EXAMPLE 10 $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{for all vectors} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$ ■

Note that the product $A\vec{x}$ is defined only if the number of *columns* of matrix A matches the number of components of vector $\vec{x}$:

$$\underbrace{\overbrace{A}^{n \times m} \overbrace{\vec{x}}^{m \times 1}}_{n \times 1}.$$

EXAMPLE 11 The product $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is undefined, because the number of columns of matrix A fails to match the number of components of vector $\vec{x}$. ■

In Definition 1.3.7, we express the product $A\vec{x}$ in terms of the *rows* of the matrix A. Alternatively, the product can be expressed in terms of the *columns*.

Let's take another look at Example 9:

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 + (-1) \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 3 \\ 1 \cdot 3 \end{bmatrix} + \begin{bmatrix} 2 \cdot 1 \\ 0 \cdot 1 \end{bmatrix} + \begin{bmatrix} 3 \cdot 2 \\ (-1) \cdot 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

We recognize that the expression $3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ involves the vect

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, the columns of A, and the scalars $x_1 = 3$, $x_3 = 2$, the components of $\vec{x}$. Thus, we can write

$$A\vec{x} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3.$$

We can generalize:

Theorem 1.3.8

> **The product $A\vec{x}$ in terms of the columns of A**
>
> If the column vectors of an $n \times m$ matrix A are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x}$ is a v $\mathbb{R}^m$ with components $x_1, \ldots, x_m$, then
>
> $$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Proof As usual, we denote the rows of A by $\vec{w}_1, \ldots, \vec{w}_n$ and the entries by a_{ij}. to show that the ith component of $A\vec{x}$ is equal to the ith component of $x_1\vec{v}$ $x_m\vec{v}_m$, for $i = 1, \ldots n$. Now

$$(i\text{th component of } A\vec{x}) \underbrace{=}_{\text{step 1}} \vec{w}_i \cdot \vec{x} = a_{i1}x_1 + \cdots a_{im}x_m$$

$$= x_1(i\text{th component of } \vec{v}_1) + \cdots$$

$$+ x_m(i\text{th component of } \vec{v}_m)$$

$$\underbrace{=}_{\text{step 4}} i\text{th component of } x_1\vec{v}_1 + \cdots + x_m\vec{v}$$

In Step 1 we are using Definition 1.3.7, and in step 4 we are using th vector addition and scalar multiplication are defined component by comp

EXAMPLE 12

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + (-4) \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix} - \begin{bmatrix} 8 \\ 20 \\ 32 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that something remarkable is happening here: Although A isn matrix and $\vec{x}$ isn't the zero vector, the product $A\vec{x}$ is the zero vector. (B the product of any two nonzero *scalars* is nonzero.)

The formula for the product $A\vec{x}$ in Theorem 1.3.8 involves the expression $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$, where $\vec{v}_1, \ldots, \vec{v}_m$ are vectors in $\mathbb{R}^n$, and $x_1, \ldots, x_m$ are scalars. Such expressions come up very frequently in linear algebra; they deserve a name.

Definition 1.3.9 Linear combinations

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ if there exist scalars $x_1, \ldots, x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

EXAMPLE 13 Is the vector $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of the vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$?

Solution

According to Definition 1.3.9, we need to see whether we can find scalars x and y such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x + 4y \\ 2x + 5y \\ 3x + 6y \end{bmatrix}$. We need to solve the linear system $\left| \begin{matrix} x + 4y = 1 \\ 2x + 5y = 1 \\ 3x + 6y = 1 \end{matrix} \right|$, with augmented matrix $M = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{bmatrix}$ and $\text{rref}(M) = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$.

It turns out that the system is consistent, with $x = -1/3$ and $y = 1/3$. The vector $\vec{b}$ is indeed a linear combination of $\vec{v}$ and $\vec{w}$, with $\vec{b} = -\frac{1}{3}\vec{v} + \frac{1}{3}\vec{w}$. ■

Note that the product $A\vec{x}$ is the linear combination of the columns of A with the components of $\vec{x}$ as the coefficients:

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Take a good look at this equation, because it is the most frequently used formula in this text. Particularly in theoretical work, it will often be useful to write the product $A\vec{x}$ as the linear combination $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$. Conversely, when dealing with a linear combination $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$, it will often be helpful to introduce the matrix

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_m \\ | & & | \end{bmatrix} \quad \text{and the vector} \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

and then write $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = A\vec{x}$.

Next we present two rules concerning the product $A\vec{x}$. In Chapter 2 we will see that these rules play a central role in linear algebra.

Theorem 1.3.10

Algebraic rules for $A\vec{x}$

If A is an $n \times m$ matrix, $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^m$, and k is a scalar, the

a. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$, and

b. $A(k\vec{x}) = k(A\vec{x})$.

We will prove the first equation, leaving the second as Exercise 45.
Denote the ith row of A by $\vec{w}_i$. Then

$$\big(i\text{th component of } A(\vec{x}+\vec{y})\big) = \vec{w}_i \cdot (\vec{x}+\vec{y}) \overset{\text{step 2}}{\widehat{=}} \vec{w}_i \cdot \vec{x} + \vec{w}_i \cdot \vec{y}$$
$$= (i\text{th component of } A\vec{x}) + (i\text{th component of } A\vec{y})$$
$$= (i\text{th component of } A\vec{x} + A\vec{y}).$$

In step 2 we are using a rule for dot products stated in Theorem A.5b, i
pendix.

Our new tools of matrix algebra allow us to see linear systems in a
as illustrated in the next example. The definition of the product $A\vec{x}$ and th
of a linear combination will be particularly helpful.

EXAMPLE 14 Consider the linear system

$$\begin{vmatrix} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{vmatrix}, \quad \text{with augmented matrix} \quad \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right].$$

We can interpret the solution of this system as the intersection of two li
x_1x_2-plane, as illustrated in Figure 2.

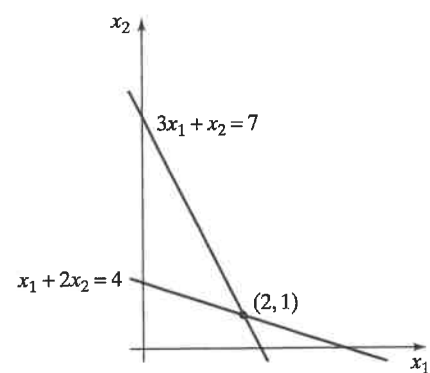


Figure 2

Alternatively, we can write the system in vector form, as

$$\begin{bmatrix} 3x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{or} \quad x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We see that solving this system amounts to writing the vector $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ as a *l*
bination of the vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. See Definition 1.3.9. The vector eq

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

and its solution can be represented geometrically, as shown in Figure 3. The problem amounts to resolving the vector $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ into two vectors parallel to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively, by means of a parallelogram.

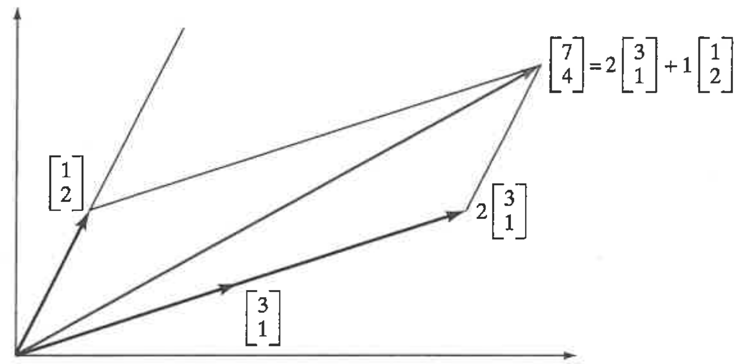


Figure 3

We can go further and write the linear combination

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{as} \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so that the linear system

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{takes the form} \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix},$$

the *matrix form* of the linear system.

Note that we started out with the augmented matrix

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right],$$

and we ended up writing the system as

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 4 \end{bmatrix}}_{\vec{b}}, \quad \text{or} \quad A\vec{x} = \vec{b}.$$

■

We can generalize:

Theorem 1.3.11 **Matrix form of a linear system**

We can write the linear system with augmented matrix $\left[A \mid \vec{b}\right]$ in matrix form as

$$A\vec{x} = \vec{b}.$$

Note that the ith component of $A\vec{x}$ is $a_{i1}x_1 + \cdots + a_{im}x_m$, by Definition 1.3.7. Thus, the ith component of the equation $A\vec{x} = \vec{b}$ is

$$a_{i1}x_1 + \cdots + a_{im}x_m = b_i;$$

this is the ith equation of the system with augmented matrix $\left[A \mid \vec{b}\right]$.

Solving the linear system $A\vec{x} = \vec{b}$ amounts to expressing vector $\vec{b}$ as a combination of the column vectors of matrix A.

EXAMPLE 15 Write the system

$$\begin{vmatrix} 2x_1 - 3x_2 + 5x_3 = 7 \\ 9x_1 + 4x_2 - 6x_3 = 8 \end{vmatrix}$$

in matrix form.

Solution

The coefficient matrix is $A = \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$. The matrix fo

$$A\vec{x} = \vec{b}, \quad \text{or} \quad \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

Now that we can write a linear system as a *single equation*, $A\vec{x} = \vec{b}$, rat a list of simultaneous equations, we can think about it in new ways.

For example, if we have an equation $ax = b$ of *numbers*, we can div sides by a to find the solution x:

$$x = \frac{b}{a} = a^{-1}b \quad (\text{if } a \neq 0).$$

It is natural to ask whether we can take an analogous approach in the ca equation $A\vec{x} = \vec{b}$. Can we "divide by A," in some sense, and write

$$\vec{x} = \frac{\vec{b}}{A} = A^{-1}\vec{b}?$$

This issue of the invertibility of a matrix will be one of the main th Chapter 2.

EXERCISES 1.3

GOAL *Use the reduced row-echelon form of the augmented matrix to find the number of solutions of a linear system. Apply the definition of the rank of a matrix. Compute the product $A\vec{x}$ in terms of the rows or the columns of A. Represent a linear system in vector or matrix form.*

1. The reduced row-echelon forms of the augmented matrices of three systems are given here. How many solutions does each system have?

a. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ **b.** $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Find the rank of the matrices in Exercises 2 through 4.

2. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

5. a. Write the system

$$\begin{vmatrix} x + 2y = 7 \\ 3x + y = 11 \end{vmatrix}$$

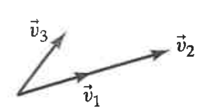
in vector form.

b. Use your answer in part (a) to represent t geometrically. Solve the system and rep solution geometrically.

6. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^2$ (sketc accompanying figure). Vectors $\vec{v}_1$ and $\vec{v}_2$ ar How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

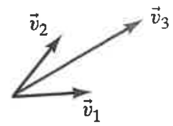
have? Argue geometrically.



7. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^2$ shown in the accompanying sketch. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

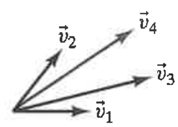
have? Argue geometrically.



8. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in $\mathbb{R}^2$ shown in the accompanying sketch. Arguing geometrically, find two solutions x, y, z of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

How do you know that this system has, in fact, infinitely many solutions?



9. Write the system

$$\left| \begin{array}{l} x + 2y + 3z = 1 \\ 4x + 5y + 6z = 4 \\ 7x + 8y + 9z = 9 \end{array} \right|$$

in matrix form.

Compute the dot products in Exercises 10 through 12 (if the products are defined).

10. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 9 & 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 13 through 15 using paper and pencil. In each case, compute the product two ways: in terms of the columns of A (Theorem 1.3.8) and in terms of the rows of A (Definition 1.3.7).

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 16 through 19 using paper and pencil (if the products are defined).

16. $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

20. a. Find $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}$.

b. Find $9 \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.

21. Use technology to compute the product

$$\begin{bmatrix} 1 & 7 & 8 & 9 \\ 1 & 2 & 9 & 1 \\ 1 & 5 & 1 & 5 \\ 1 & 6 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 5 \\ 6 \end{bmatrix}.$$

22. Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

23. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

24. Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

25. Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ is inconsistent. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

26. Let A be a 4×3 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

27. If the rank of a 4×4 matrix A is 4, what is rref(A)?

28. If the rank of a 5×3 matrix A is 3, what is rref(A)?

In Problems 29 through 32, let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ *and* $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

29. Find a *diagonal* matrix A such that $A\vec{x} = \vec{y}$.

30. Find a matrix A of rank 1 such that $A\vec{x} = \vec{y}$.

31. Find an *upper triangular* matrix A such that $A\vec{x} = \vec{y}$, where all the entries of A on and above the diagonal are nonzero.

32. Find a matrix A with all nonzero entries such that $A\vec{x} = \vec{y}$.

33. Let A be the $n \times n$ matrix with all 1's on the diagonal and all 0's above and below the diagonal. What is $A\vec{x}$, where $\vec{x}$ is a vector in $\mathbb{R}^n$?

34. We define the vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$.

a. For

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},$$

compute $A\vec{e}_1$, $A\vec{e}_2$, and $A\vec{e}_3$.

b. If B is an $n \times 3$ matrix with columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, what are $B\vec{e}_1$, $B\vec{e}_2$, and $B\vec{e}_3$?

35. In $\mathbb{R}^m$, we define

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th component.}$$

If A is an $n \times m$ matrix, what is $A\vec{e}_i$?

36. Find a 3×3 matrix A such that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

$$\text{and} \quad A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

37. Find all vectors $\vec{x}$ such that $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

38. a. Using technology, generate a random 3×3 matrix A. (The entries may be either single-digit integers or numbers between 0 and 1, depending on the technology you are using.) Find rref(A). Repeat this experiment a few times.

b. What does the reduced row-echelon form of most 3×3 matrices look like? Explain.

39. Repeat Exercise 38 for 3×4 matrices.

40. Repeat Exercise 38 for 4×3 matrices.

41. How many solutions do most systems of th equations with three unknowns have? Explai of your work in Exercise 38.

42. How many solutions do most systems of th equations with four unknowns have? Explai of your work in Exercise 39.

43. How many solutions do most systems of f equations with three unknowns have? Explai of your work in Exercise 40.

44. Consider an $n \times m$ matrix A with more columns ($n > m$). Show that there is a vect such that the system $A\vec{x} = \vec{b}$ is inconsistent.

45. Consider an $n \times m$ matrix A, a vector $\vec{x}$ in scalar k. Show that

$$A(k\vec{x}) = k(A\vec{x}).$$

46. Find the rank of the matrix

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},$$

where a, d, and f are nonzero, and b, c, and trary numbers.

47. A linear system of the form

$$A\vec{x} = \vec{0}$$

is called *homogeneous*. Justify the following

a. All homogeneous systems are consistent

b. A homogeneous system with fewer equ unknowns has infinitely many solutions.

c. If $\vec{x}_1$ and $\vec{x}_2$ are solutions of the homog tem $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_2$ is a solution

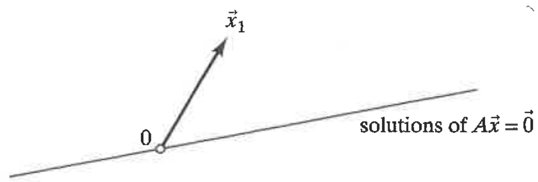
d. If $\vec{x}$ is a solution of the homogene $A\vec{x} = \vec{0}$ and k is an arbitrary consta is a solution as well.

48. Consider a solution $\vec{x}_1$ of the linear syste Justify the facts stated in parts (a) and (b):

a. If $\vec{x}_h$ is a solution of the system $A\vec{x}$ $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x}$

b. If $\vec{x}_2$ is another solution of the system A $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x}$ =

c. Now suppose A is a 2×2 matrix. A sol $\vec{x}_1$ of the system $A\vec{x} = \vec{b}$ is shown in panying figure. We are told that the solu system $A\vec{x} = \vec{0}$ form the line shown in Draw the line consisting of all solution tem $A\vec{x} = \vec{b}$.



If you are puzzled by the generality of this problem, think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

49. Consider the accompanying table. For some linear systems $A\vec{x} = \vec{b}$, you are given either the rank of the coefficient matrix A, or the rank of the augmented matrix $\left[A \mid \vec{b}\right]$. In each case, state whether the system could have no solution, one solution, or infinitely many solutions. There may be more than one possibility for some systems. Justify your answers.

	Number of Equations	**Number of Unknowns**	**Rank of A**	**Rank of $\left[A \mid \vec{b}\right]$**
a.	3	4	—	2
b.	4	3	3	—
c.	4	3	—	4
d.	3	4	3	—

50. Consider a linear system $A\vec{x} = \vec{b}$, where A is a 4×3 matrix. We are told that rank $\left[A \mid \vec{b}\right] = 4$. How many solutions does this system have?

51. Consider an $n \times m$ matrix A, an $r \times s$ matrix B, and a vector $\vec{x}$ in $\mathbb{R}^p$. For which values of n, m, r, s, and p is the product

$$A(B\vec{x})$$

defined?

52. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Can you find a 2×2 matrix C such that

$$A(B\vec{x}) = C\vec{x},$$

for all vectors $\vec{x}$ in $\mathbb{R}^2$?

53. If A and B are two $n \times m$ matrices, is

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}$$

for all $\vec{x}$ in $\mathbb{R}^m$?

54. Consider two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$ that are not parallel. Which vectors in $\mathbb{R}^3$ are linear combinations of $\vec{v}_1$ and $\vec{v}_2$? Describe the set of these vectors geometrically. Include a sketch in your answer.

55. Is the vector $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ a linear combination of

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}?$$

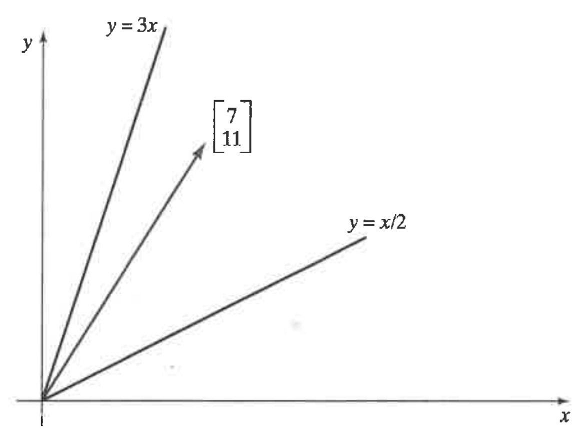
56. Is the vector

$$\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$$

a linear combination of

$$\begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}?$$

57. Express the vector $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ as the sum of a vector on the line $y = 3x$ and a vector on the line $y = x/2$.



58. For which values of the constants b and c is the vector

$$\begin{bmatrix} 3 \\ b \\ c \end{bmatrix} \text{ a linear combination of } \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}?$$

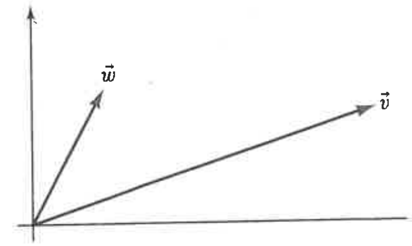
59. For which values of the constants c and d is $\begin{bmatrix}5\\7\\c\\d\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$?

60. For which values of the constants a, b, c, and d is $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ a linear combination of $\begin{bmatrix}0\\0\\3\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\4\\0\end{bmatrix}$, and $\begin{bmatrix}2\\0\\5\\6\end{bmatrix}$?

61. For which values of the constant c is $\begin{bmatrix}1\\c\\c^2\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\2\\4\end{bmatrix}$ and $\begin{bmatrix}1\\3\\9\end{bmatrix}$?

62. For which values of the constant c is $\begin{bmatrix}1\\c\\c^2\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\a\\a^2\end{bmatrix}$ and $\begin{bmatrix}1\\b\\b^2\end{bmatrix}$, where a and b are arbitrary constants?

In Exercises 63 through 68, consider the vectors $\vec{v}$ and $\vec{w}$ in the accompanying figure.



63. Give a geometrical description of the set of a of the form $\vec{v} + c\vec{w}$, where c is an arbitrary rea

64. Give a geometrical description of the set of a of the form $\vec{v} + c\vec{w}$, where $0 \le c \le 1$.

65. Give a geometrical description of the set of a of the form $a\vec{v} + b\vec{w}$, where $0 \le a \le 1$ and 0

66. Give a geometrical description of the set of a of the form $a\vec{v} + b\vec{w}$, where $a + b = 1$.

67. Give a geometrical description of the set of a of the form $a\vec{v}+b\vec{w}$, where $0 \le a$, $0 \le b$, and

68. Give a geometrical description of the set of $\vec{u}$ in $\mathbb{R}^2$ such that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$.

69. Solve the linear system

$$\begin{vmatrix} & y + z = a \\ x & + z = b \\ x + y & = c \end{vmatrix},$$

where a, b, and c are arbitrary constants.

70. Let A be the $n \times n$ matrix with 0's on the m nal, and 1's everywhere else. For an arbitra in $\mathbb{R}^n$, solve the linear system $A\vec{x} = \vec{b}$, exp components $x_1, \dots, x_n$ of $\vec{x}$ in terms of the c of $\vec{b}$. See Exercise 69 for the case $n = 3$.

Chapter One Exercises

TRUE OR FALSE?[19]

Determine whether the statements that follow are true or false, and justify your answer.

1. If A is an $n \times n$ matrix and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the product $A\vec{x}$ is a linear combination of the columns of matrix A.

2. If vector $\vec{u}$ is a linear combination of vec $\vec{w}$, then we can write $\vec{u} = a\vec{v} + b\vec{w}$ for sc a and b.

3. Matrix $\begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ is in reduced row-echel

4. A system of four linear equations in three u always inconsistent.

5. There exists a 3×4 matrix with rank 4.

6. If A is a 3×4 matrix and vector $\vec{v}$ is in $\mathbb{R}^4$, $A\vec{v}$ is in $\mathbb{R}^3$.

7. If the 4×4 matrix A has rank 4, then any li with coefficient matrix A will have a uniqu

[19] We will conclude each chapter (except for Chapter 9) with some true–false questions, over 400 in all. We will start with a group of about 10 straightforward statements that refer directly to definitions and theorems given in the chapter. Then there may be some computational exercises, and the remaining ones are more conceptual, calling for independent reasoning. In some chapters, a few of the problems toward the end can be quite challenging. Don't expect a balanced coverage of all the topics; some concepts are better suited for this kind of questioning than others.