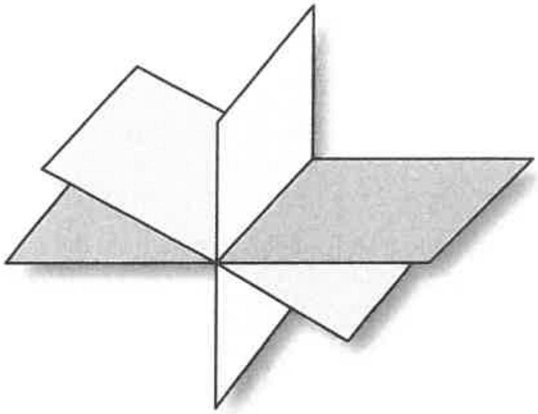

CHAPTER

1

Linear Equations

1.1 Introduction to Linear Systems

Traditionally, algebra was the art of solving equations and systems of equations. The word *algebra* comes from the Arabic *al-jabr* (الجبر), which means *restoration* (of broken parts).[1] The term was first used in a mathematical sense by Mohammed al-Khowarizmi (c. 780–850), who worked at the House of Wisdom, an academy established by Caliph al-Ma'mun in Baghdad. Linear algebra, then, is the art of solving systems of linear equations.

The need to solve systems of linear equations frequently arises in mathematics, statistics, physics, astronomy, engineering, computer science, and economics.

Solving systems of linear equations is not conceptually difficult. For small systems, ad hoc methods certainly suffice. Larger systems, however, require more systematic methods. The approach generally used today was beautifully explained 2,000 years ago in a Chinese text, the *Nine Chapters on the Mathematical Art* (Jiuzhang Suanshu, 九章算術).[2] Chapter 8 of that text, called *Method of Rectangular Arrays* (Fang Cheng, 方程), contains the following problem:

> The yield of one bundle of inferior rice, two bundles of medium-grade rice, and three bundles of superior rice is 39 *dou* of grain.[3] The yield of one bundle of inferior rice, three bundles of medium-grade rice, and two bundles of superior rice is 34 *dou*. The yield of three bundles of inferior rice, two bundles of medium-grade rice, and one bundle of superior rice is 26 *dou*. What is the yield of one bundle of each grade of rice?

In this problem the unknown quantities are the yields of one bundle of inferior, one bundle of medium-grade, and one bundle of superior rice. Let us denote these quantities by x, y, and z, respectively. The problem can then be represented by the

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[1] At one time, it was not unusual to see the sign *Algebrista y Sangrador* (bone setter and blood letter) at the entrance of a Spanish barber's shop.

[2] Shen Kangshen et al. (ed.), *The Nine Chapters on the Mathematical Art*, Companion and Commentary, Oxford University Press, 1999.

[3] The *dou* is a measure of volume, corresponding to about 2 liters at that time.

following system of linear equations:

$$\left| \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array} \right| .$$

To solve for x, y, and z, we need to transform this system from the form

$$\left| \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array} \right| \quad \text{into the form} \quad \left| \begin{array}{lll} x & & = \ldots \\ & y & = \ldots \\ & & z = \ldots \end{array} \right| .$$

In other words, we need to eliminate the terms that are off the diagonal, those in the following equations, and make the coefficients of the variables alc diagonal equal to 1:

$$\begin{array}{rcrcrcr} x & + & \textcircled{2y} & + & \textcircled{3z} & = & 39 \\ \textcircled{x} & + & 3y & + & \textcircled{2z} & = & 34 \\ \textcircled{3x} & + & \textcircled{2y} & + & z & = & 26. \end{array}$$

We can accomplish these goals step by step, one variable at a time. In tl you may have simplified systems of equations by adding equations to one ; or subtracting them. In this system, we can eliminate the variable x from the equation by subtracting the first equation from the second:

$$\left| \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array} \right| \quad \begin{array}{c} \longrightarrow \\ -1\text{st equation} \\ \end{array} \quad \left| \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ 3x & + & 2y & + & z & = & 26 \end{array} \right| .$$

To eliminate the variable x from the third equation, we subtract the first e from the third equation three times. We multiply the first equation by 3 to g

$$3x + 6y + 9z = 117 \qquad (3 \times 1\text{st equation})$$

and then subtract this result from the third equation:

$$\left| \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ 3x & + & 2y & + & z & = & 26 \end{array} \right| \quad \begin{array}{c} \longrightarrow \\ \\ -3 \times 1\text{st equation} \end{array} \quad \left| \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ & - & 4y & - & 8z & = & -91 \end{array} \right.$$

Similarly, we eliminate the variable y above and below the diagonal:

$$\left| \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ & - & 4y & - & 8z & = & -91 \end{array} \right| \quad \begin{array}{c} -2 \times 2\text{nd equation} \\ \longrightarrow \\ +4 \times 2\text{nd equation} \end{array} \quad \left| \begin{array}{rcrcrcr} x & & & + & 5z & = & 49 \\ & & y & - & z & = & -5 \\ & & & - & 12z & = & -111 \end{array} \right.$$

Before we eliminate the variable z above the diagonal, we make the coeffic on the diagonal equal to 1, by dividing the last equation by -12:

$$\left| \begin{array}{rcrcrcr} x & & & + & 5z & = & 49 \\ & & y & - & z & = & -5 \\ & & & - & 12z & = & -111 \end{array} \right| \quad \begin{array}{c} \\ \longrightarrow \\ \div(-12) \end{array} \quad \left| \begin{array}{rcrcrcr} x & & & + & 5z & = & 49 \\ & & y & - & z & = & -5 \\ & & & & z & = & 9.25 \end{array} \right| .$$

Finally, we eliminate the variable z above the diagonal:

$$\left| \begin{array}{rcrcrcr} x & & & + & 5z & = & 49 \\ & & y & - & z & = & -5 \\ & & & & z & = & 9.25 \end{array} \right| \quad \begin{array}{c} -5 \times \text{third equation} \\ + \text{third equation} \\ \longrightarrow \end{array} \quad \left| \begin{array}{lll} x & & = 2.75 \\ & y & = 4.25 \\ & & z = 9.25 \end{array} \right| .$$

The yields of inferior, medium-grade, and superior rice are 2.75, 4.25, and ' per bundle, respectively.

By substituting these values, we can check that $x = 2.75$, $y = 4.25$, $z = 9.25$ is indeed the solution of the system:

$$\begin{aligned} 2.75 + 2 \times 4.25 + 3 \times 9.25 &= 39 \\ 2.75 + 3 \times 4.25 + 2 \times 9.25 &= 34 \\ 3 \times 2.75 + 2 \times 4.25 + \quad 9.25 &= 26. \end{aligned}$$

Happily, in linear algebra, you are almost always able to check your solutions. It will help you if you get into the habit of checking now.

Geometric Interpretation

How can we interpret this result geometrically? Each of the three equations of the system defines a plane in x–y–z-space. The solution set of the system consists of those points (x, y, z) that lie in all three planes (i.e., the intersection of the three planes). Algebraically speaking, the solution set consists of those ordered triples of numbers (x, y, z) that satisfy all three equations simultaneously. Our computations show that the system has only one solution, $(x, y, z) = (2.75, 4.25, 9.25)$. This means that the planes defined by the three equations intersect at the point $(x, y, z) = (2.75, 4.25, 9.25)$, as shown in Figure 1.

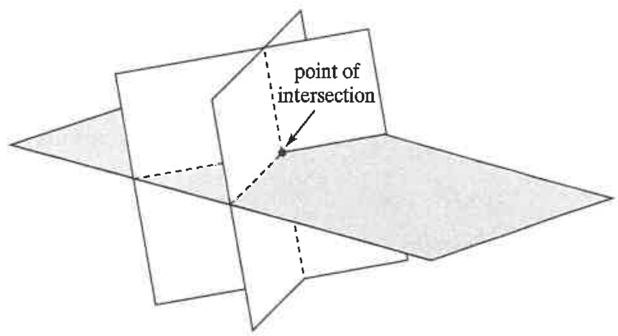


Figure 1 Three planes in space, intersecting at a point.

While three different planes in space usually intersect at a point, they may have a line in common (see Figure 2a) or may not have a common intersection at all, as shown in Figure 2b. Therefore, a system of three equations with three unknowns may have a unique solution, infinitely many solutions, or no solutions at all.

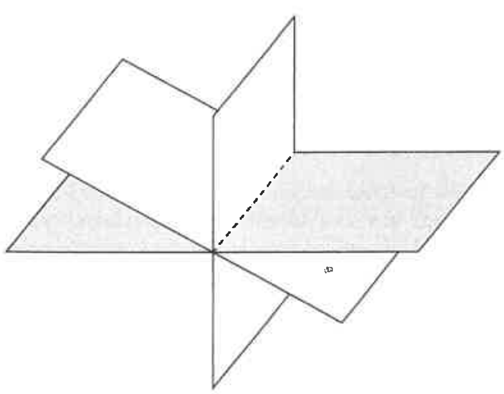


Figure 2(a) Three planes having a line in common.

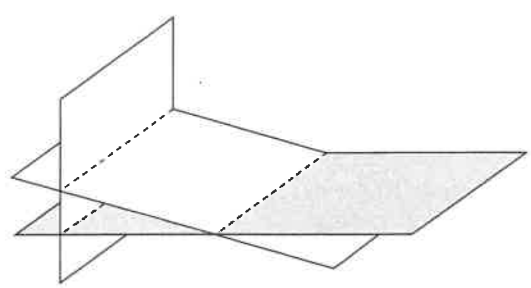


Figure 2(b) Three planes with no common intersection.

A System with Infinitely Many Solutions

Next, let's consider a system of linear equations that has infinitely many solu

$$\begin{vmatrix} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{vmatrix}.$$

We can solve this system using the method of elimination as previous cussed. For simplicity, we label the equations with Roman numerals.

$$\begin{vmatrix} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{vmatrix} \begin{matrix} \div 2 \\ \longrightarrow \\ \end{matrix} \begin{vmatrix} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{vmatrix} \begin{matrix} \longrightarrow \\ -4\,(\mathrm{I}) \\ -7\,(\mathrm{I}) \end{matrix}$$

$$\begin{vmatrix} x + 2y + 3z = 0 \\ -3y - 6z = 3 \\ -6y - 12z = 6 \end{vmatrix} \begin{matrix} \longrightarrow \\ \div(-3) \\ \end{matrix} \begin{vmatrix} x + 2y + 3z = 0 \\ y + 2z = -1 \\ -6y - 12z = 6 \end{vmatrix} \begin{matrix} -2\,(\mathrm{II}) \\ \longrightarrow \\ +6\,(\mathrm{II}) \end{matrix}$$

$$\begin{vmatrix} x - z = 2 \\ y + 2z = -1 \\ 0 = 0 \end{vmatrix} \longrightarrow \begin{vmatrix} x - z = 2 \\ y + 2z = -1 \end{vmatrix}$$

After omitting the trivial equation $0 = 0$, we are left with only two eq with three unknowns. The solution set is the intersection of two nonparallel in space (i.e., a line). This system has infinitely many solutions.

The two foregoing equations can be written as follows:

$$\begin{vmatrix} x = z + 2 \\ y = -2z - 1 \end{vmatrix}.$$

We see that both x and y are determined by z. We can freely choose a value arbitrary real number; then the two preceding equations give us the values o y for this choice of z. For example,

- Choose $z = 1$. Then $x = z + 2 = 3$ and $y = -2z - 1 = -3$. The sol $(x, y, z) = (3, -3, 1)$.
- Choose $z = 7$. Then $x = z + 2 = 9$ and $y = -2z - 1 = -15$. The s is $(x, y, z) = (9, -15, 7)$.

More generally, if we choose $z = t$, an arbitrary real number, we get x and $y = -2t - 1$. Therefore, the general solution is

$$(x, y, z) = (t + 2, -2t - 1, t) = (2, -1, 0) + t(1, -2, 1).$$

This equation represents a line in space, as shown in Figure 3.

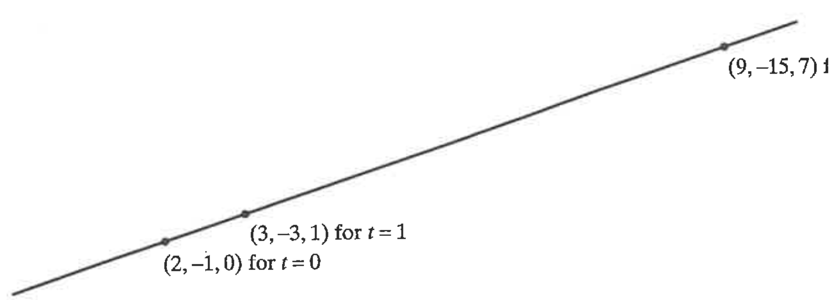


Figure 3 The line $(x, y, z) = (t + 2, -2t - 1, t)$.

A System without Solutions

In the following system, perform the eliminations yourself to obtain the result shown:

$$\left|\begin{array}{l} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{array}\right| \longrightarrow \left|\begin{array}{rcl} x \quad - z &=& 2 \\ y + 2z &=& -1 \\ 0 &=& -6 \end{array}\right|.$$

Whatever values we choose for x, y, and z, the equation $0 = -6$ cannot be satisfied. This system is *inconsistent*; that is, it has no solutions.

EXERCISES 1.1

GOAL ***Set up and solve systems with as many as three linear equations with three unknowns, and interpret the equations and their solutions geometrically.***

In Exercises 1 through 10, find all solutions of the linear systems using elimination as discussed in this section. Then check your solutions.

1. $\left|\begin{array}{l} x + 2y = 1 \\ 2x + 3y = 1 \end{array}\right|$

2. $\left|\begin{array}{l} 4x + 3y = 2 \\ 7x + 5y = 3 \end{array}\right|$

3. $\left|\begin{array}{l} 2x + 4y = 3 \\ 3x + 6y = 2 \end{array}\right|$

4. $\left|\begin{array}{l} 2x + 4y = 2 \\ 3x + 6y = 3 \end{array}\right|$

5. $\left|\begin{array}{l} 2x + 3y = 0 \\ 4x + 5y = 0 \end{array}\right|$

6. $\left|\begin{array}{l} x + 2y + 3z = 8 \\ x + 3y + 3z = 10 \\ x + 2y + 4z = 9 \end{array}\right|$

7. $\left|\begin{array}{l} x + 2y + 3z = 1 \\ x + 3y + 4z = 3 \\ x + 4y + 5z = 4 \end{array}\right|$

8. $\left|\begin{array}{l} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 10z = 0 \end{array}\right|$

9. $\left|\begin{array}{l} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 \\ 7x + 2y - 3z = 1 \end{array}\right|$

10. $\left|\begin{array}{l} x + 2y + 3z = 1 \\ 2x + 4y + 7z = 2 \\ 3x + 7y + 11z = 8 \end{array}\right|$

In Exercises 11 through 13, find all solutions of the linear systems. Represent your solutions graphically, as intersections of lines in the x–y-plane.

11. $\left|\begin{array}{l} x - 2y = 2 \\ 3x + 5y = 17 \end{array}\right|$

12. $\left|\begin{array}{l} x - 2y = 3 \\ 2x - 4y = 6 \end{array}\right|$

13. $\left|\begin{array}{l} x - 2y = 3 \\ 2x - 4y = 8 \end{array}\right|$

In Exercises 14 through 16, find all solutions of the linear systems. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

14. $\left|\begin{array}{l} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{array}\right|$

15. $\left|\begin{array}{l} x + y - z = 0 \\ 4x - y + 5z = 0 \\ 6x + y + 4z = 0 \end{array}\right|$

16. $\left|\begin{array}{l} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 0 \end{array}\right|$

17. Find all solutions of the linear system

$$\left|\begin{array}{l} x + 2y = a \\ 3x + 5y = b \end{array}\right|,$$

where a and b are arbitrary constants.

18. Find all solutions of the linear system

$$\left|\begin{array}{l} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{array}\right|,$$

where a, b, and c are arbitrary constants.

19. Consider the linear system

$$\left|\begin{array}{l} x + y - z = -2 \\ 3x - 5y + 13z = 18 \\ x - 2y + 5z = k \end{array}\right|,$$

where k is an arbitrary number.

a. For which value(s) of k does this system have one or infinitely many solutions?

b. For each value of k you found in part a, how many solutions does the system have?

c. Find all solutions for each value of k.

20. Consider the linear system

$$\left|\begin{array}{l} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (k^2 - 5)z = k \end{array}\right|,$$

where k is an arbitrary constant. For which value(s) of k does this system have a unique solution? For which value(s) of k does the system have infinitely many solutions? For which value(s) of k is the system inconsistent?

21. The sums of any two of three real numbers are 24, 28, and 30. Find these three numbers.

22. Emile and Gertrude are brother and sister. Emile has twice as many sisters as brothers, and Gertrude has just as many brothers as sisters. How many children are there in this family?

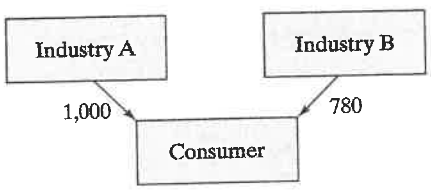
23. Consider a two-commodity market. When the unit prices of the products are P_1 and P_2, the quantities demanded, D_1 and D_2, and the quantities supplied, S_1 and S_2, are given by

$$\begin{aligned} D_1 &= 70 - 2P_1 + P_2, & S_1 &= -14 + 3P_1, \\ D_2 &= 105 + P_1 - P_2, & S_2 &= -7 \qquad + 2P_2. \end{aligned}$$

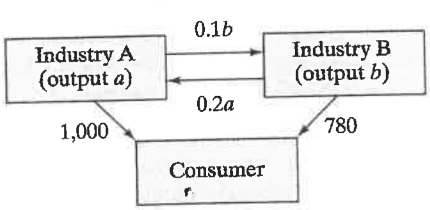
a. What is the relationship between the two commodities? Do they compete, as do Volvos and BMWs, or do they complement one another, as do shirts and ties?

b. Find the equilibrium prices (i.e., the prices for which supply equals demand), for both products.

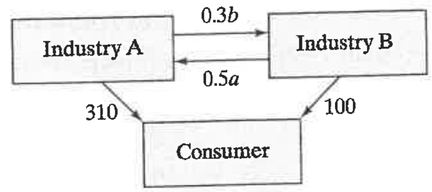
24. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906–1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input–output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1,000 and 780, in millions of dollars per year.



What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand? You may be tempted to say 1,000 and 780, respectively, but things are not quite as simple as that. We have to take into account the interindustry demand as well. Let us say that industry A produces electricity. Of course, producing almost any product will require electric power. Suppose that industry B needs 10¢ worth of electricity for each \$1 of output B produces and that industry A needs 20¢ worth of B's products for each \$1 of output A produces. Find the outputs a and b needed to satisfy both consumer and interindustry demand.



25. Find the outputs a and b needed to satisfy the consumer and interindustry demands given in the following figure. See Exercise 24:



26. Consider the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t).$$

This equation could describe a forced damped os tor, as we will see in Chapter 9. We are told th differential equation has a solution of the form

$$x(t) = a\sin(t) + b\cos(t).$$

Find a and b, and graph the solution.

27. Find all solutions of the system

$$\left| \begin{aligned} 7x - \ \ y &= \lambda x \\ -6x + 8y &= \lambda y \end{aligned} \right|, \qquad \text{for}$$

a. $\lambda = 5$ **b.** $\lambda = 10$, and **c.** $\lambda = 1$[...]

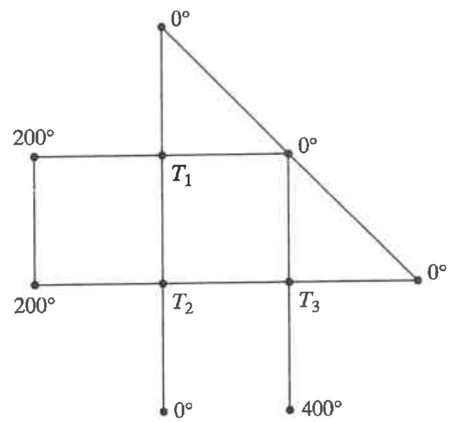
28. On a sunny summer day, you are taking the boat ride from Stein am Rhein, Switzerla Schaffhausen, down the Rhein River. This nonst takes 40 minutes, but the return trip to Stein, ups will take a full hour. Back in Stein, you decide on the boat and continue on to Constance, Ge now traveling on the still waters of Lake Con How long will this nonstop trip from Stein to Co take? You may assume that the boat is traveli constant speed relative to the water throughout a the Rhein River flows at a constant speed betwee and Schaffhausen. The traveling distance from S and Schaffhausen is the same as from Stein to Const

29. On your next trip to Switzerland, you should t scenic boat ride from Rheinfall to Rheinau an The trip downstream from Rheinfall to Rheina 20 minutes, and the return trip takes 40 minu distance between Rheinfall and Rheinau along t is 8 kilometers. How fast does the boat travel (to the water), and how fast does the river Rhe in this area? You may assume both speeds to stant throughout the journey.

30. In a grid of wires, the temperature at exteri points is maintained at constant values (in °C), a in the accompanying figure. When the grid is in equilibrium, the temperature T at each interi point is the average of the temperatures at adjacent points. For example,

$$T_2 = \frac{T_3 + T_1 + 200 + 0}{4}.$$

Find the temperatures T_1, T_2, and T_3 when the grid is in thermal equilibrium.



31. Find the polynomial of degree 2 [a polynomial of the form $f(t) = a + bt + ct^2$] whose graph goes through the points $(1, -1)$, $(2, 3)$, and $(3, 13)$. Sketch the graph of this polynomial.

32. Find a polynomial of degree ≤ 2 [of the form $f(t) = a + bt + ct^2$] whose graph goes through the points $(1, p)$, $(2, q)$, $(3, r)$, where p, q, r are arbitrary constants. Does such a polynomial exist for all values of p, q, r?

33. Find all the polynomials $f(t)$ of degree ≤ 2 [of the form $f(t) = a + bt + ct^2$] whose graphs run through the points $(1, 3)$ and $(2, 6)$, such that $f'(1) = 1$ [where $f'(t)$ denotes the derivative].

34. Find all the polynomials $f(t)$ of degree ≤ 2 [of the form $f(t) = a + bt + ct^2$] whose graphs run through the points $(1, 1)$ and $(2, 0)$, such that $\int_1^2 f(t)\,dt = -1$.

35. Find all the polynomials $f(t)$ of degree ≤ 2 [of the form $f(t) = a + bt + ct^2$] whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 1$.

36. Find all the polynomials $f(t)$ of degree ≤ 2 [of the form $f(t) = a + bt + ct^2$] whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 3$.

37. Find the function $f(t)$ of the form $f(t) = ae^{3t} + be^{2t}$ such that $f(0) = 1$ and $f'(0) = 4$.

38. Find the function $f(t)$ of the form $f(t) = a\cos(2t) + b\sin(2t)$ such that $f''(t) + 2f'(t) + 3f(t) = 17\cos(2t)$. (This is the kind of differential equation you might have to solve when dealing with forced damped oscillators, in physics or engineering.)

39. Find the circle that runs through the points $(5, 5)$, $(4, 6)$, and $(6, 2)$. Write your equation in the form $a + bx + cy + x^2 + y^2 = 0$. Find the center and radius of this circle.

40. Find the ellipse centered at the origin that runs through the points $(1, 2)$, $(2, 2)$, and $(3, 1)$. Write your equation in the form $ax^2 + bxy + cy^2 = 1$.

41. Find all points (a, b, c) in space for which the system

$$\left| \begin{aligned} x + 2y + 3z &= a \\ 4x + 5y + 6z &= b \\ 7x + 8y + 9z &= c \end{aligned} \right|$$

has at least one solution.

42. Linear systems are particularly easy to solve when they are in triangular form (i.e., all entries above or below the diagonal are zero).

a. Solve the lower triangular system

$$\left| \begin{aligned} x_1 \qquad\qquad\qquad\qquad &= -3 \\ -3x_1 + x_2 \qquad\qquad\quad &= 14 \\ x_1 + 2x_2 + x_3 \qquad &= 9 \\ -x_1 + 8x_2 - 5x_3 + x_4 &= 33 \end{aligned} \right|$$

by forward substitution, finding x_1 first, then x_2, then x_3, and finally x_4.

b. Solve the upper triangular system

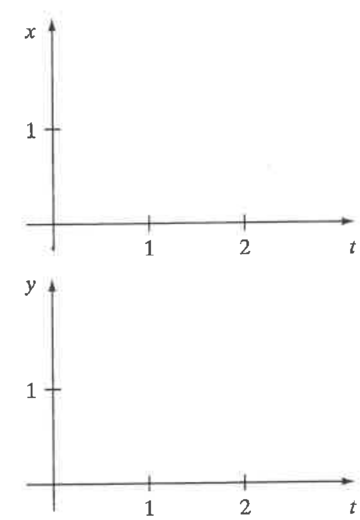
$$\left| \begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= -3 \\ x_2 + 3x_3 + 7x_4 &= 5 \\ x_3 + 2x_4 &= 2 \\ x_4 &= 0 \end{aligned} \right|.$$

43. Consider the linear system

$$\left| \begin{aligned} x + y &= 1 \\ x + \frac{t}{2}y &= t \end{aligned} \right|,$$

where t is a nonzero constant.

a. Determine the x- and y-intercepts of the lines $x + y = 1$ and $x + (t/2)y = t$; sketch these lines. For which values of the constant t do these lines intersect? For these values of t, the point of intersection (x, y) depends on the choice of the constant t; that is, we can consider x and y as functions of t. Draw rough sketches of these functions.



Explain briefly how you found these graphs. Argue geometrically, without solving the system algebraically.

b. Now solve the system algebraically. Verify that the graphs you sketched in part (a) are compatible with your algebraic solution.

44. Find a system of linear equations with three unknowns whose solutions are the points on the line through (1, 1, 1) and (3, 5, 0).

45. Find a system of linear equations with three unknowns x, y, z whose solutions are

$$x = 6 + 5t, \quad y = 4 + 3t, \quad \text{and} \quad z = 2 + t,$$

where t is an arbitrary constant.

46. Boris and Marina are shopping for chocolate bars. Boris observes, "If I add half my money to yours, it will be enough to buy two chocolate bars." Marina naively asks, "If I add half my money to yours, how many can we buy?" Boris replies, "One chocolate bar." How much money did Boris have? (From Yuri Chernyak and Robert Rose, *The Chicken from Minsk*, Basic Books, 1995.)

47. Here is another method to solve a system of linear equations: Solve one of the equations for one of the variables, and substitute the result into the other equations. Repeat this process until you run out of variables or equations. Consider the example discussed on page 2:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array} \right| .$$

We can solve the first equation for x:

$$x = 39 - 2y - 3z.$$

Then we substitute this equation into the other equations:

$$\left| \begin{array}{l} (39 - 2y - 3z) + 3y + 2z = 34 \\ 3(39 - 2y - 3z) + 2y + z = 26 \end{array} \right| .$$

We can simplify:

$$\left| \begin{array}{l} y - z = -5 \\ -4y - 8z = -91 \end{array} \right| .$$

Now, $y = z - 5$, so that $-4(z - 5) - 8z = -$

$$-12z = -111.$$

We find that $z = \dfrac{111}{12} = 9.25$. Then

$$y = z - 5 = 4.25,$$

and

$$x = 39 - 2y - 3z = 2.75.$$

Explain why this method is essentially the sa
method discussed in this section; only the boo
is different.

48. A hermit eats only two kinds of food: brown
yogurt. The rice contains 3 grams of protei
grams of carbohydrates per serving, while th
contains 12 grams of protein and 20 grams of
drates.

a. If the hermit wants to take in 60 grams o
and 300 grams of carbohydrates per day, h
servings of each item should he consume?

b. If the hermit wants to take in P grams o
and C grams of carbohydrates per day, h
servings of each item should he consume?

49. I have 32 bills in my wallet, in the denomin
US$ 1, 5, and 10, worth \$100 in total. How m
have of each denomination?

50. Some parking meters in Milan, Italy, accept co
denominations of 20¢, 50¢, and €2. As an
program, the city administrators offer a big r
brand new Ferrari Testarossa) to any meter m
brings back exactly 1,000 coins worth exactly
from the daily rounds. What are the odds of th
being claimed anytime soon?

1.2 Matrices, Vectors, and Gauss–Jordan Elimination

When mathematicians in ancient China had to solve a system of simultaneo
equations such as[4]

$$\left| \begin{array}{l} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{array} \right| ,$$

[4]This example is taken from Chapter 8 of the *Nine Chapters on the Mathematical Art*; see p
source is George Gheverghese Joseph, *The Crest of the Peacock, Non-European Roots of Ma*
3rd ed., Princeton University Press, 2010.