Group Quiz 3
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Math 153

Name: Key

No work = no credit

1.) Find the length of the curve C described by $\vec{r}(t) = \langle 8\cos t + 8t\sin t, 8\sin t - 8t\cos t \rangle$ on $0 \le t \le \pi/2$

$x'(t) = \cancel{-8s} + \cancel{8s} + 8tc$

$y'(t) = \cancel{8c} - \cancel{8c} + 8ts$

$$L = \int_0^{\pi/2} \sqrt{(8t\cos t)^2 + (8t\sin t)^2}\, dt$$

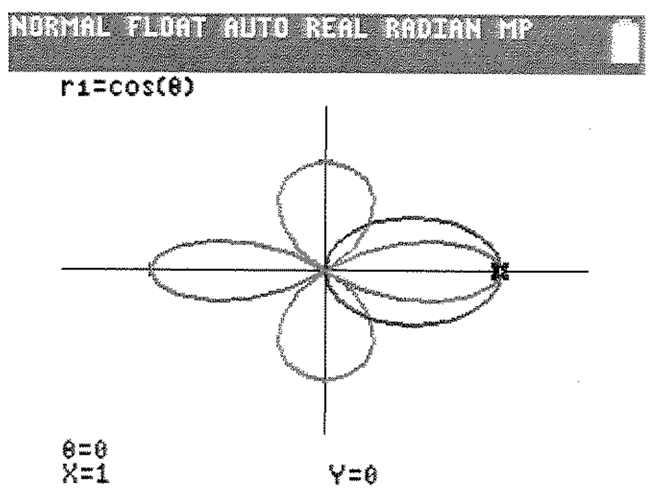
$$= \int_0^{\pi/2} \sqrt{64t^2}\, dt$$

$$= \int_0^{\pi/2} 8t\, dt$$

$$= \left[4t^2\right]_0^{\pi/2}$$

$$= \pi^2$$

The graph to the right relates to questions (2.) and (3.) on the next page where you will be asked to find the angle between the curves at the point of intersection and the area enclosed by both curves.



2.) Find the acute angle between curves $r_1 = \cos\theta$ and $r_2 = \cos 2\theta$ at their intersection point in the first quadrant.

1st: Find θ at intersection points

solve $\cos\theta = \cos 2\theta$

$\Rightarrow 0 = 2c^2 - c - 1$

$\Rightarrow 0 = (2c+1)(c-1)$

$\Rightarrow \cos\theta = -\frac{1}{2}$ OR $\cos\theta = 1$

$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0, \pi$

$\uparrow$ 1st quadrant.

$\boxed{\text{2 alternate step 3's are attached!!}}$

2nd: Find $\frac{dy}{dx}$ @ the intersection point.

$r_1: \frac{dy}{dx} = \frac{-s\cdot s + c\cdot c}{-s\cdot c - c\cdot s} = \frac{+\cos 2\theta}{-\sin 2\theta}$

$\rightarrow \frac{dy}{dx} = -\cot(2\theta)\Big|_{\theta = \frac{4\pi}{3}} \quad \frac{\sqrt{3}}{3}$

$r_2: \frac{dy}{dx} = \frac{-2\sin 2\theta \cdot \sin\theta + \cos 2\theta \cos\theta}{-2\sin 2\theta \cos\theta - \cos 2\theta \sin\theta}$

$= \frac{-4s^2c + (2c^2-1)c}{-4sc^2 - (1-2s^2)s}$

$= \frac{-4(1-c^2)c + 2c^3 - c}{-4s(1-s^2) - s + 2s^3}$

$= \frac{-4c + 4c^3 + 2c^3 - c}{-4s + 4s^3 - s + 2s^3}$

$= \frac{6c^3 - 5c}{6s^3 - 5s}\Big|_{\theta = \frac{4\pi}{3}} \quad \frac{7\sqrt{3}}{3}$

3rd: construct vectors w/ the right slope.

$\vec{r}_1 = \langle 3, \sqrt{3}\rangle$ and $\vec{r}_2 = \langle 3, 7\sqrt{3}\rangle$

4th: Find the angle α between vectors.

$\alpha = \cos^{-1}\left(\frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|}\right)$

$= \cos^{-1}\left(\frac{30}{\sqrt{12}\cdot\sqrt{156}}\right)$

≈ 0.80 rad OR $46.1°$

3.) Find the exact area inside the curves $r_1 = \cos\theta$ and $r_2 = \cos 2\theta$.

Area = 2 +

= 2 + 2 + 2

$= 2\int_{\frac{4\pi}{3}}^{\frac{3\pi}{2}} \frac{1}{2} r_1^2 \, d\theta + 2\int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} \frac{1}{2} r_2^2 \, d\theta + 2\int_0^{\frac{\pi}{4}} \frac{1}{2} r_2^2 \, d\theta$

$= \int_{\frac{4\pi}{3}}^{\frac{3\pi}{2}} \cos^2\theta \, d\theta + \int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} \cos^2 2\theta \, d\theta + \int_0^{\pi/4} \cos^2 2\theta \, d\theta$

$= \int_{\frac{4\pi}{3}}^{3\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta + \int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} \frac{1+\cos 4\theta}{2} \, d\theta + \int_0^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2} \, d\theta$

$= \left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_{\frac{4\pi}{3}}^{3\pi/2} + \left[\frac{1}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\right]_{\frac{5\pi}{4}}^{4\pi/3} + \left[\frac{1}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\right]_0^{\pi/4}$

$= \frac{\pi}{4} + \frac{1}{4}\left[\sin 2\theta\right]_{4\pi/3}^{3\pi/2} + \frac{1}{8}\left[\sin 4\theta\right]_{\frac{5\pi}{4}}^{4\pi/3} + \frac{1}{8}\left[\sin 4\theta\right]_0^{\pi/4}$

$= \frac{\pi}{4} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{8}\cdot\frac{-\sqrt{3}}{2} + 0$

$= \frac{\pi}{\;} - \frac{3\sqrt{3}}{\;}$

Alternate solutions.

(1) $m_1 = \frac{\sqrt{3}}{3}$

$m_2 = \frac{7\sqrt{3}}{3}$

so the direction angles w/ the x-axis (see section 12.3) are $\arctan\left(\frac{\sqrt{3}}{3}\right)$ and $\arctan\left(\frac{7\sqrt{3}}{3}\right)$ respectively.

The angle between the tangent lines is the difference $\tan^{-1}\left(\frac{7\sqrt{3}}{3}\right) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \approx 46.1°$.

(2) Given slopes m_1 & m_2 (neither undefined), the acute angle between lines is

$$\alpha = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$

In this case, $\alpha = \tan^{-1}\left|\frac{\frac{7\sqrt{3}}{3} - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}\cdot\frac{7\sqrt{3}}{3}}\right|$

$$= \tan^{-1}\left|\frac{3\sqrt{3}}{5}\right|$$

$$\approx 46.1°.$$