13.4: Motion in Space: Velocity and Acceleration

If $\vec{r}(t)$ gives the position of a particle moving thru space, then (as expected) $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{r}\,''(t)$.

Ex 1: Find and sketch $\vec{r}, \vec{v}, \vec{a}$ at $t = \pi/6$ if

$\vec{r}(t) = \langle \sin(t), 2\cos(t) \rangle$.

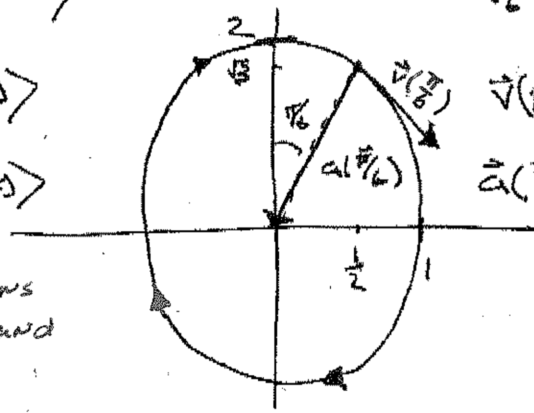
$\vec{v}(t) = \langle \cos(t), -2\sin(t) \rangle$

$\vec{a}(t) = \langle -\sin(t), -2\cos(t) \rangle$

$\vec{r}(\frac{\pi}{6}) = \langle \frac{1}{2}, \sqrt{3} \rangle$

$\vec{v}(\frac{\pi}{6}) = \langle \frac{\sqrt{3}}{2}, -1 \rangle$

$\vec{a}(\pi/6) = \langle -\frac{1}{2}, -\sqrt{3} \rangle$



main point of ex 1: $\vec{r}, \vec{v}$, and $\vec{a}$ have unique directions and $\vec{a}$ has a tangential and normal component.

Ex 2: Find $\vec{v}(t)$ and $\vec{r}(t)$ if $\vec{a}(t) = \langle 0, 0, 1 \rangle$ and $\vec{v}(0) = \langle 1, -1, 2 \rangle$ and $\vec{r}(2) = \vec{0}$.

main point of ex 2: Integration requires finding constants of integration.

Tangential and Normal components of Acceleration

$\vec{v}$ gives velocity and we use $v = |\vec{v}|$ to represent speed.

Goal: Find a_T and a_N s.t. $\vec{a} = a_T \vec{T} + a_N \vec{N}$

Now, recall $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\vec{v}}{v}$

This is the derivation.

$\Rightarrow \vec{v} = v\vec{T}$

$\Rightarrow \vec{a} = \vec{v}\,' = v'\vec{T} + v\vec{T}\,'$

This is almost of the form $\vec{a} = a_T\vec{T} + a_N\vec{N}$, but we must relate $\vec{T}\,'$ and $\vec{N}$.

To do this, we incorporate curvature. Recall $k = \dfrac{|\vec{T}\,'|}{|\vec{r}\,'|} = \dfrac{|\vec{T}\,'|}{v} \Rightarrow |\vec{T}\,'| = kv$.

but we defined the unit normal as $\vec{N} = \dfrac{\vec{T}\,'}{|\vec{T}\,'|}$

$\Rightarrow \vec{T}\,' = \vec{N}|\vec{T}\,'| = kv\vec{N}$

$*$ Hence $\vec{a} = v'\vec{T} + kv^2\vec{N}$

Now we can write

$$\vec{a} = a_T \vec{T} + a_N \vec{N} \quad \text{where } a_T = v' \text{ and } a_N = k v^2$$

Find a_T and a_N in terms of $\vec{r}$.

a_T: $\vec{v} \cdot \vec{a} = v T \cdot (v' \vec{T} + k v^2 \vec{N})$

$= v v' \vec{T} \cdot \vec{T} + k v^3 \vec{T} \cdot \vec{N}$ (since $\vec{T} \cdot \vec{T} = 1$, $\vec{T} \cdot \vec{N} = 0$)

$= v \cdot v'$ solve $\vec{v} \cdot \vec{a} = v v'$ for v'

emphasize that these are scalars

$$\Rightarrow a_T = v' = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$$

a_N: $$k v^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} |\vec{r}'(t)|^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$$

<u>Ex 3:</u> A particle moves along the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

$$\boxed{\text{Notice } \quad k = \frac{a_N}{|\vec{r}'(t)|^2}}$$

<u>cool observation</u>

$a_T \neq 0$ and $a_N = 0$ corresponds to linear motion.

$a_T = 0$ and $a_N \neq 0$ corresponds to circular motion.