12.5: Equations of Lines and Planes

LINES

Recall from Friday (animation), that a line thru the point given by position vector $\vec{r}_0$ and in the direction of $\vec{v}$, can be expressed by the vector equation $\vec{r} = \vec{r}_0 + t\cdot\vec{v}$ (t a parameter).

If $\vec{r} = \langle x, y, z\rangle$

$\vec{r}_0 = \langle x_0, y_0, z_0\rangle$

$v = \langle a, b, c\rangle$, then.

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

and we have the parametric equations of a line L thru (x_0, y_0, z_0) parallel to $\vec{v}$

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc, \quad t \in \mathbb{R}$$

Ex1: Find the vector equation of the line thru $(1,2,3)$ parallel to $\langle 4,5,6\rangle$

Eliminating the parameter gives the symmetric equations

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \quad \text{when } a,b,c \neq 0.$$

If (for example) $a=0$, $x = x_0$, $\frac{y-y_0}{b} = \frac{z-z_0}{c}$ which is a line on the plane $x = x_0$.

<u>Ex2</u>: Find the symmetric equations of the line in (ex1) & the pt where it intersects the xy-plane, ← (when $z=0$).

To describe the line from $\vec{r}_0$ to $\vec{r}_1$, we have

$\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \le t \le 1$. (don't memorize).

PLANES

Planes can be ~~determined~~ by a point and a normal vector (orthogonal). why?

If $\vec{r}$ and $\vec{r}_0$ are position vectors of points on the plane w/ normal $\vec{n}$, then $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$.

This can be written as $\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$ (vector equation of the plane).

If $\vec{n} = \langle a,b,c\rangle$

$\vec{r} = \langle x,y,z\rangle$

and $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, then

$$\vec{n}\cdot(\vec{r}-\vec{r}_0) = \langle a,b,c\rangle\cdot\langle x-x_0, y-y_0, z-z_0\rangle$$
$$= a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

<u>Ex3</u>: Find the equation of the plane thru $(1,2,3)$ w/ normal $\vec{n}=\langle 4,5,6\rangle$

Question: How would I find the equation of the plane thru 3 points that are not colinear?

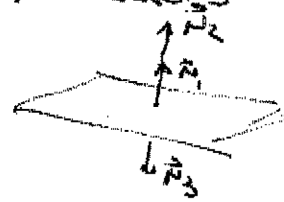
Defn. We define the angle between planes to be the angle between normals.

Ex4: Find the parametric equations for the line of intersection of the planes $z = x + y$ & $2x - 5y - z = 1$

a) use $\vec{n}_1 \times \vec{n}_2$ to find $\vec{v}$ // to the line L.

b) Let $z = 0$ to find a point on L.

NOTE: There are an infinite number of vectors normal to a plane at a point



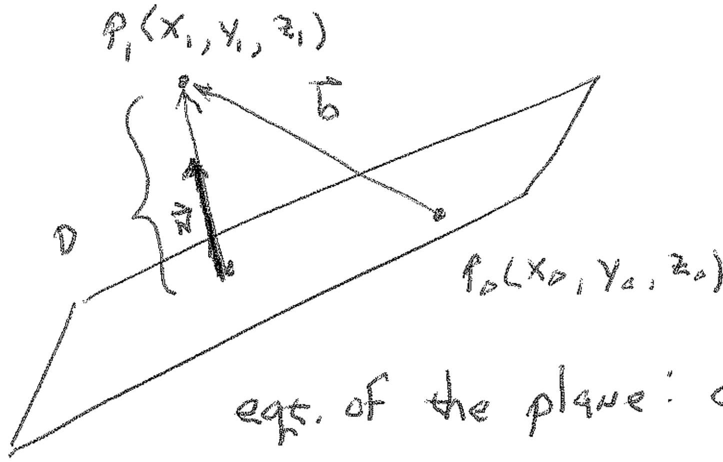
See derivation on the next page.

ex: Find the distance from the point $(1, -2, 4)$ to the plane $3x + 2y + 6z = 5$

Soln: $$D = \frac{|3(1) + 2(-2) + 6(4) + (-5)|}{\sqrt{3^2 + 2^2 + 6^2}}$$

$$= \frac{18}{7}$$

Distance from a plane to a point.



eqt. of the plane: $ax + by + cz + d = 0$

Note: D is the length of a normal vector but not necessarily the magnitude of $\langle a, b, c \rangle$ as there are an infinite number of normal vectors.

$$D = |\mathrm{comp}_{\vec{n}} \vec{b}| \quad \leftarrow \text{absolute value.}$$

$$= \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} \quad \leftarrow \text{magnitude.}$$

$$= \frac{|\langle a, b, c \rangle \cdot \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle|}{|\vec{n}|}$$

$$= \frac{|ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0|}{|\vec{n}|}$$

$$= \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{|\vec{n}|}$$

$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

since $P_0(x_0, y_0, z_0)$ is on the plane, $ax_0 + by_0 + cz_0 + d = 0$