

# 11.5: Alternating Series

examples of alternating series form

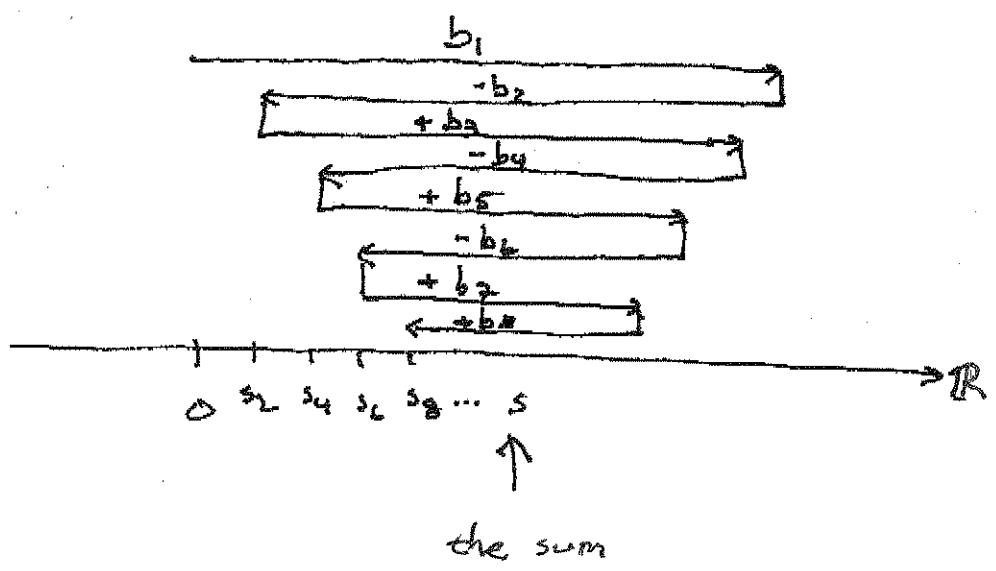
i)  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$   $(-1)^{n+1} b_n, b_n > 0$

ii)  $-1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \dots$   $(-1)^n b_n, b_n > 0$

## The alternating series test

If  $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - \dots$  ( $b_n > 0$ )  
 satisfies the conditions  $b_{n+1} \leq b_n \forall n$  and  $\lim_{n \rightarrow \infty} b_n = 0$ , then the series converges.  
magnitude decreasing (monotonic)  
decreases to zero

## The picture



□ proof.

Recall from 11.1 #72 that if  $\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} a_{2n+1} = L$

that  $\lim_{n \rightarrow \infty} a_n = L$ .

so, notice  $s_2 = b_1 - b_2 \geq 0$

$$s_4 = s_2 + (b_3 - b_4) \geq s_2$$

$$s_6 = s_4 + (b_5 - b_6) \geq s_4$$

⋮

$$s_{2n} = s_{2n-2} + ( \quad ) \geq s_{2n-2}$$

so,  $\{s_{2n}\}$  is increasing.

bounded above



now  $s_{2n} = b_1 - (b_2 - b_3) - \dots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1$

so  $\{s_{2n}\}$  is increasing and bounded above, hence it converges, and  $\lim_{n \rightarrow \infty} s_{2n} = s$ .

$$\begin{aligned} \text{Now } \lim_{n \rightarrow \infty} s_{2n+1} &= \lim_{n \rightarrow \infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n \rightarrow \infty} s_{2n} + \lim_{n \rightarrow \infty} b_{2n+1} \\ &= s + 0 \\ &= s \end{aligned}$$

since the even or odd partial sums converge

to  $s$ ,  $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = s$ . □

Ex1: Test for convergence:  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Ex2: Test for convergence:  $\sum_{n=1}^{\infty} (-1)^{n+1}$ .

Q: What do I do if I can't easily see that  $\{b_n\}$  is decreasing?

Q: What if the series begins by increasing?

### Estimating Sums (Friendly)

From the picture, and proof, the sum  $S$  always lives between  $S_N$  and  $S_{N+1}$  so

$$|S - S_N| \leq |S_{N+1} - S_N| = b_{N+1}. \text{ So if } S = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

is the sum of an alternating series and

$0 \leq b_{N+1} \leq b_N$  and  $\lim_{n \rightarrow \infty} b_n = 0$ , then

$$|R_N| = |S - S_N| \leq b_{N+1}.$$

Ex4: Approx  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  to 4 places.