11.2: Series

We call the sum of a sequence a series.

So, $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \dots$

As may be expected we will attempt to evaluate the series by making use of the limit $\sum_{i=1}^{\infty} a_i = \lim_{N\to\infty} \sum_{i=1}^{N} a_i$

Ex1: $\sum_{i=1}^{\infty} i = 1+2+3+\dots = \lim_{N\to\infty} \sum_{i=1}^{N} i = \lim_{N\to\infty} \frac{N(N+1)}{2} = \infty$

Ex2: $\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \lim_{N\to\infty} \sum_{i=0}^{N} \left(\frac{1}{2}\right)^i = \lim_{N\to\infty} 2 - \left(\frac{1}{N}\right) = 2.$

Defn: If the N^{th} partial sum $S_N = \sum_{i=1}^{N} a_i$, then
$\sum_{i=1}^{\infty} a_i = \lim_{N\to\infty} \sum_{i=1}^{N} a_i = \lim_{N\to\infty} S_N.$

If $\{S_N\}$ is convergent to S, then $\sum_{i=1}^{\infty} a_i = S$.
If the partial sums diverge, then the series diverges.

The Geometric Series.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots = \lim_{m\to\infty} \sum_{n=1}^{m} ar^{n-1}$$

Now $S_m = \sum_{n=1}^{m} ar^{n-1}$

$$= a + ar + ar^2 + \dots + ar^{m-1}$$

$$= (a + ar + \dots + ar^{m-1})\left(\frac{1-r}{1-r}\right)$$

$$= \frac{a + ar + \dots + ar^{m-1} - ar - \dots - ar^{m-1} - ar^m}{1-r}$$

$$= \frac{a - ar^m}{1-r}$$

$$= \frac{a(1-r^m)}{1-r}$$

So, $\sum_{n=1}^{\infty} ar^{n-1} = \lim_{m\to\infty} \frac{a(1-r^m)}{1-r} = a \cdot \lim_{m\to\infty} \frac{1-r^m}{1-r}$

When does this sequence converge?

$r > 1$

$r = 1$

$-1 < r < 1$

$r < -1$

So, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $|r| < 1$, else the series diverges.

Ex3: Find the sum of the geometric sequence. $\sum_{n=1}^{\infty} 3^{n-4}\, 2^{4-4n}$.

Ex4: Find the sum of the geometric sequence $\sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n$.

Ex5: Rationalize $3.14\overline{15}1515$.

Ex6: Find the domain of $f(x) = \sum_{n=1}^{\infty} x^n$.

Ex7: $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ (partial fractions, telescoping series).

Ex8: Show the harmonic series diverges.

$s_1 = 1$

$s_2 = 1 + \frac{1}{2}$

$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{2}{2}$

$s_8 = 1 + \frac{1}{2} + (\quad) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{3}{2}$

etc and $s_{2^n} > 1 + \frac{n}{2}$.

<u>Thm</u>: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

$\square$ proof

Let $s_n = a_1 + a_2 + \dots + a_n$.

Now $s_n - s_{n-1} = a_n$.

but the series converges to s and so.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

<u>Note</u>: w/ $\sum_1^{} a_n$ we have the associated sequences $\{a_n\}$ and $\{s_n\}$.

<u>Note</u>: $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges (counterexample).

<u>corollary</u>: If $\lim_{n\to\infty} a_n$ D.N.E. or is non-zero, then $\sum_1^{} a_n$ diverges.

<u>Ex 9</u>: Test $\sum_1^{} \frac{18n^4}{(3n+1)(n-2)}$ for convergence.

Question: What does $\lim_{n\to\infty} a_n = 0$ imply regarding $\sum a_n$?

Thm: If $\sum_1 a_n$ and $\sum_1 b_n$ are convergent & c is a constant, then...

i) $\sum_1 c a_n = c \sum_1 a_n$

ii) $\sum (a_n \pm b_n) = \sum_1 a_n \pm \sum_1 b_n$

(prove if time using partial sums.)

Ex 10: $\sum_{n=0}^{\infty} \left[\frac{2}{n^2+4n+3} + \left(\frac{1}{2}\right)^n \right]$

(note that the series begins at zero).