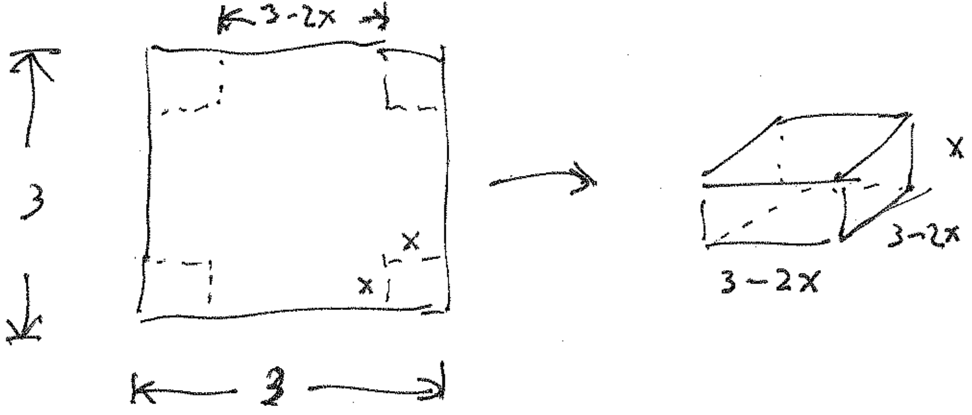
4.7
1/7

Optimization

Example 1: A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(1) Draw a pic.



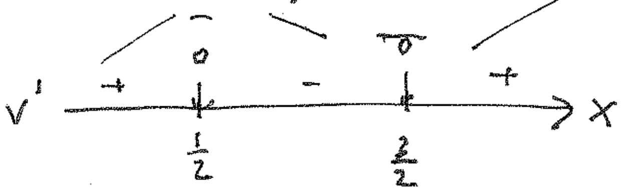
(2) set up an eqt.

$$v(x) = x(3-2x)^2$$
$$= x(9-12x+4x^2)$$
$$= 4x^3-12x^2+9x$$

(3) Take the deriv.

$$v'(x) = 12x^2-24x+9$$
$$= 3(4x^2-8x+3)$$
$$= 3(2x-1)(2x-3)$$

(4) sign diagram

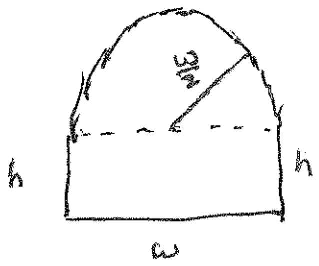


Max at $(\frac{1}{2}, 2)$

(5) The max volume is 2 ft^3.

Example 2: A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 feet, find the dimensions of the window so that the greatest possible amount of light is admitted.

(1) Draw a pic



(2) Set up an eqt.

perimeter: $2h + w + \pi \cdot \frac{w}{2} = 30 \Rightarrow h = \frac{30 - \frac{\pi \cdot w}{2} - w}{2}$

area: $A = hw + \frac{\pi \cdot (\frac{w}{2})^2}{2}$

$$= \frac{30 - \frac{\pi \cdot w}{2} - w}{2} \cdot w + \frac{\pi w^2}{8}$$

$$= 15w - \frac{\pi w^2}{4} - \frac{w^2}{2} + \frac{\pi w^2}{8}$$

$$= 15w + w^2\left(\frac{-\pi}{4} - \frac{1}{2} + \frac{\pi}{8}\right)$$

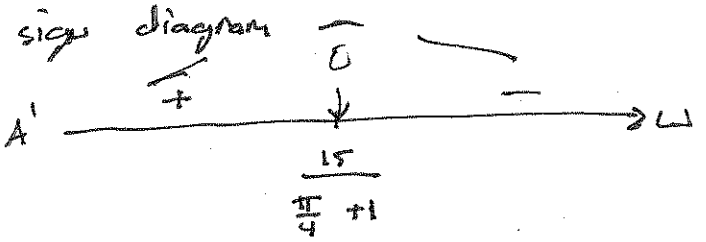
$$= 15w + \left(-\frac{\pi}{8} - \frac{1}{2}\right)w^2$$

(3) Take deriv.

$A' = 15 - 2\left(\frac{\pi}{8} + \frac{1}{2}\right)w$

zero @ $w = \frac{15}{2\left(\frac{\pi}{8} + \frac{1}{2}\right)}$

(4) sign diagram



max @ $w = \frac{15}{\frac{\pi}{4} + 1} \approx 8.40$

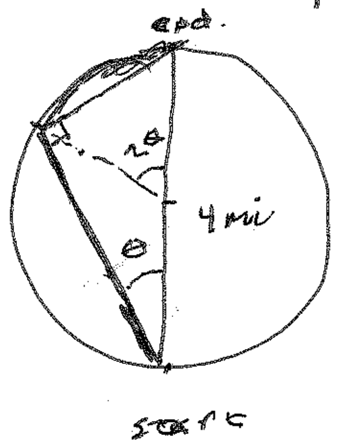
(5) Answer.

The optimal window has width 8.4ft and height of 4.2ft.

<u>Solution 1</u>: using right triangle trig.

<u>Example 3</u>: A woman is at the boat launch of a circular lake with radius 2 miles that has a sidewalk all the way around it. She wants to arrive at the point diametrically opposite the launch (on the other side of the lake) in the shortest possible time. She can walk at the rate of 4 mph and row a boat at 2mph. How should she proceed?

(1) draw a pic



boat only ↓ walk only ↙

note: $0 \le \theta \le \frac{\pi}{2}$

distance by boat $= 4\cos\theta$

distance by foot $= 2\cdot 2\theta = 4\theta$

(2) set up eqt.

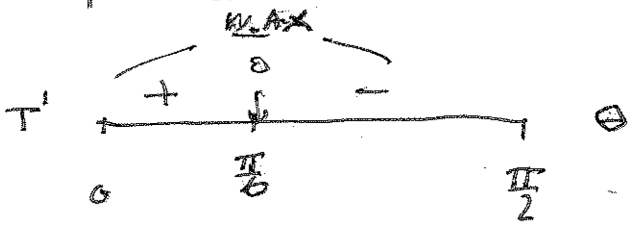
$$T(\theta) = \frac{4\cos\theta}{2} + \frac{4\theta}{4}$$

$$= 2\cos(\theta) + \theta$$

(3) differentiate

$T'(\theta) = 1 - 2\sin\theta$

(4) optimize



check endpoints: $T(0) = 2$

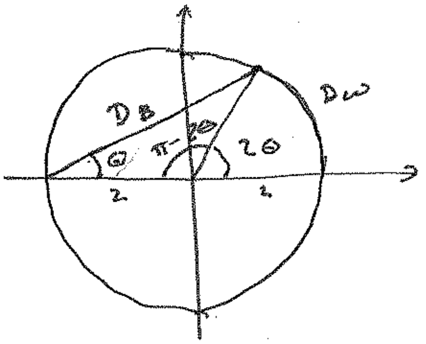
$T(\frac{\pi}{2}) = \pi/2$ min.

(5) Ans: the question

she should walk.

Solution 2: using the law of cosines

Example 3: A woman is at the boat launch of a circular lake with radius 2 miles that has a sidewalk all the way around it. She wants to arrive at the point diametrically opposite the launch (on the other side of the lake) in the shortest possible time. She can walk at the rate of 4 mph and row a boat at 2mph. How should she proceed?



$$D_w = 2\cdot 2\theta$$

$$D_B = \sqrt{8-8\cos(\pi-2\theta)}$$

$$= 2\sqrt{2-2\cos(\pi-2\theta)}$$

and $0 \le \theta \le \frac{\pi}{2}$

So $$T(\theta) = \frac{D_w}{4} + \frac{D_B}{2}$$

$$= \theta + \sqrt{2-2\cos(\pi-2\theta)}$$

$$\Rightarrow T'(\theta) = 1 + \frac{+\cancel{2}\sin(\pi-2\theta)(-2)}{\cancel{2}\sqrt{2-2\cos(\pi-2\theta)}}$$

$$= 1 - \frac{2\sin(\pi-2\theta)}{\sqrt{2-2\cos(\pi-2\theta)}}$$

and solve $T'(\theta)=0$

$$\frac{1}{2} = \frac{\sin(\pi-2\theta)}{\sqrt{2-2\cos(\pi-2\theta)}}$$

$$\Rightarrow \frac{1}{4} = \frac{\sin^2(\pi-2\theta)}{2-2\cos(\pi-2\theta)}$$

$$\Rightarrow \frac{1}{4}(2-2\cos(\pi-2\theta)) = 1-\cos^2(\pi-2\theta)$$

$$\Rightarrow \cos^2(\pi-2\theta) - \frac{1}{2}\cos(\pi-2\theta) - \frac{1}{2} = 0$$

$$\Rightarrow 2c^2 - c - 1 = 0 \quad \text{where } c=\cos(\pi-2\theta)$$

$$\Rightarrow (2c+1)(c-1)=0$$

$$\Rightarrow c=-\frac{1}{2} \text{ or } c=1$$

So $\cos(\pi-2\theta)=1$

$\Rightarrow \pi-2\theta = 0 \Rightarrow \theta=\frac{\pi}{2}$

or

$\pi-2\theta = 2\pi \Rightarrow \theta=-\frac{\pi}{2}$
(outside domain)

or $\cos(\pi-2\theta) = -\frac{1}{2}$

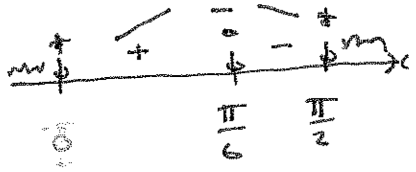
$\Rightarrow \pi-2\theta = \frac{2\pi}{3} \Rightarrow \theta=\frac{\pi}{6}$

or

$\pi-2\theta = \frac{4\pi}{3} \Rightarrow \theta=-\frac{\pi}{6}$
(outside domain)

- But T' is undefined when $\cos(\pi-2\theta)=1$ or $\theta=\pm\frac{\pi}{2}$
- Did we pick up other extraneous solutions?

$T'(\frac{\pi}{6})=0$ & $T'(-\frac{\pi}{6})=2$
(extra soln)



There is a max time when $\theta=\frac{\pi}{6}$

hmmm.

check endpoints.

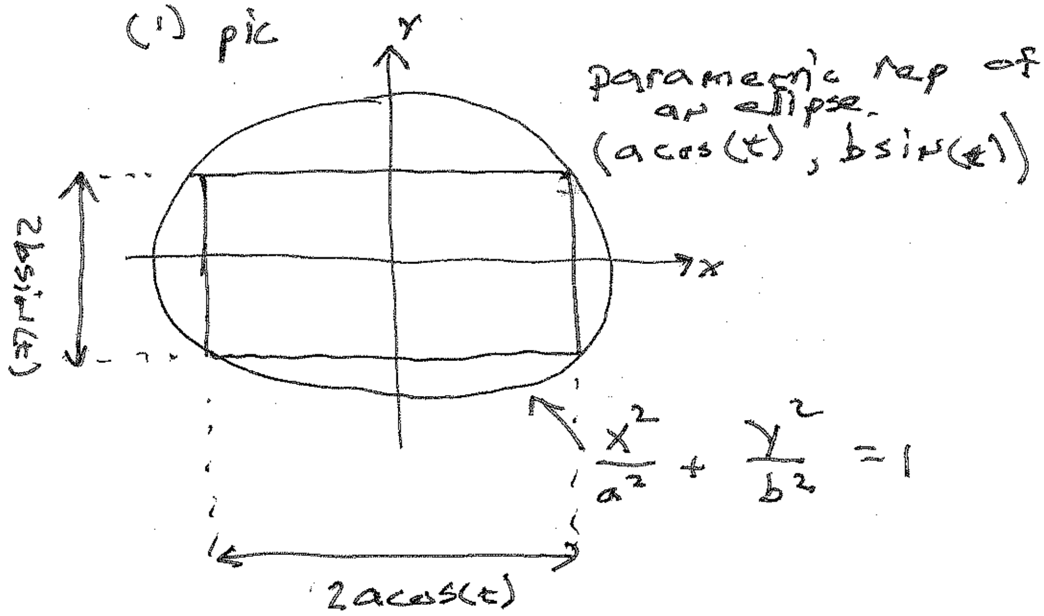
$T(0) = 2$ hrs; $T(\frac{\pi}{2})\approx 1.5$

She should walk

Solution 1: using parametric equations

Example 4: Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(1) pic



(2) eqt. for area.

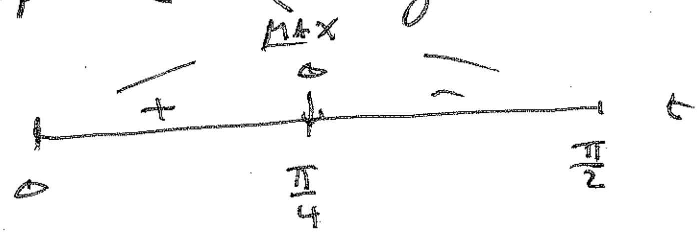
$A(t) = 2a\cos(t)\cdot 2b\sin(t)$

$= 2ab\sin(2t)$

(3) differentiate

$A'(t) = 4ab\cos(2t)$

(4) optimize (only consider $0 \le t \le \pi/2$)



and $A(\frac{\pi}{4}) = 2ab$

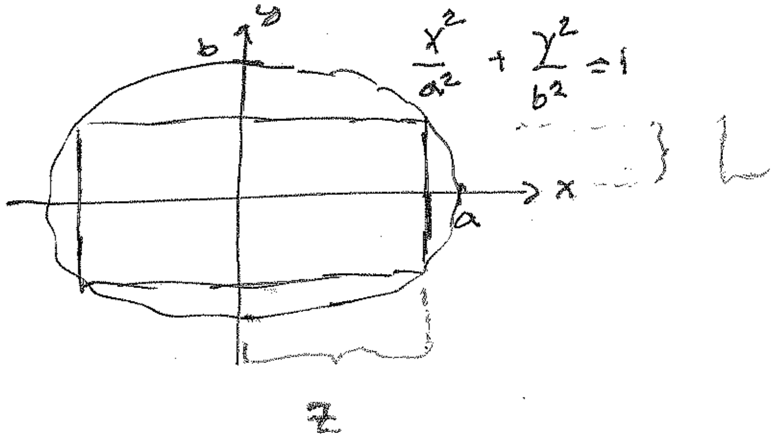
(5) Ans. the question

The max area is 2ab.

Solution 2 : Using Rectangular Coordinates

Example 4: Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(1) draw a pic.



(2) set up eqt.

$$\text{Area} = (2z)(2L) = 4zL$$

and $L = \pm \frac{b}{a}\sqrt{a^2-z^2}$

so $A = 4z \cdot \frac{b}{a}\sqrt{a^2-z^2}$

(3) differentiate.

$$A' = \frac{4b}{a}\left(1 \cdot \sqrt{a^2-z^2} + z \cdot \frac{-\cancel{2}z}{\cancel{2}\sqrt{a^2-z^2}}\right)$$

$$= \frac{4b}{a} \cdot \frac{(a^2-z^2) - z^2}{\sqrt{a^2-z^2}}$$

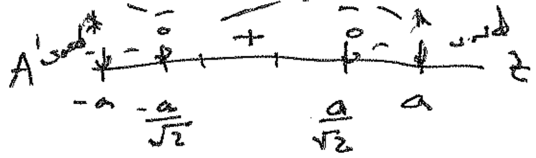
$$= \frac{4b}{a} \cdot \frac{a^2-2z^2}{\sqrt{a^2-z^2}}$$

(4) find zeros $(A'=0)$.

solve $a^2 - 2z^2 = 0$

$$\Rightarrow \frac{a^2}{2} = z^2$$

$$\Rightarrow z = \pm \frac{a}{\sqrt{2}}$$



(5) Maximize area

$$z = \frac{a}{\sqrt{2}}$$

$$A = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{a}\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}$$

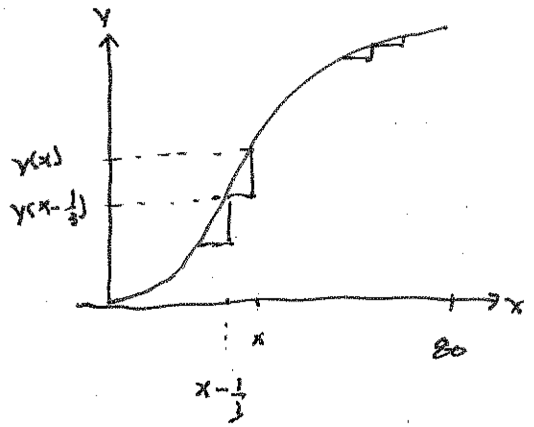
$$= \frac{4b}{\sqrt{2}}\sqrt{\frac{a^2}{2}}$$

$$= \frac{4ab}{2}$$

solution 1: using a difference

Example 5: A contractor is engaged to build steps up the slope of a hill that has the shape of the graph $y = \frac{x^2(120-x)}{6400}, 0 \le x \le 80$ with x in meters. What is the maximum vertical rise of a stair if each stair has a horizontal length of one-third meter?

(1) pic



stair ht fct $h(x) = y(x) - y(x-\frac{1}{3})$

this is valid for $x = \frac{1}{3}, \frac{2}{3}, \ldots, 80$

(2) eqt:

$$h(x) = y(x) - y(x-\tfrac{1}{3})$$
$$= \frac{1}{6400}\left(-x^2 + \frac{2169}{27}x - \frac{361}{27}\right)$$

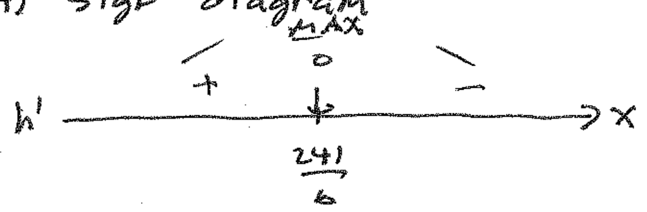
(I did this in Mathematica)

(3) differentiate

$$h'(x) = \frac{1}{6400}\left(-2x + \frac{2169}{27}\right)$$

$h'(x) = 0$ when $x = \frac{241}{6}$

(4) sign diagram



The max is at $\frac{241}{6}$ which isn't in the domain. So we check the nearest values 40 & $40\frac{1}{3}$.

$$\rightarrow h(40) = h(40\tfrac{1}{3}) = \frac{43199}{172800}$$
$$\approx 0.249994$$

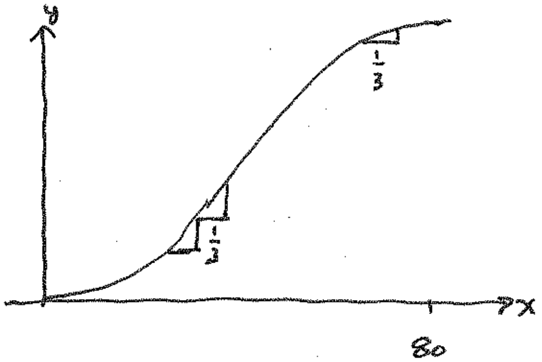
(5) conclusion.

The max stair height is about $\frac{1}{4}$ m.

solution 2: using linearization

Example 5: A contractor is engaged to build steps up the slope of a hill that has the shape of the graph $y = \dfrac{x^2(120-x)}{6400}, 0 \le x \le 80$ with *x* in meters. What is the maximum vertical rise of a stair if each stair has a horizontal length of one-third meter?

(1) pic



observe: the stair ht is most where the hill is steepest.

y : hill ht

y' : hill steepness

y'' : find max hill steepness.

(2) eqt.

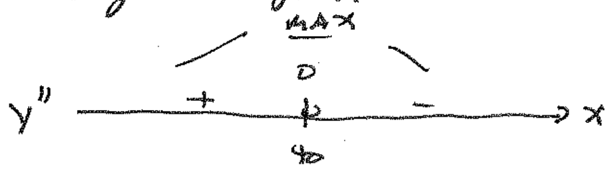
$$y = \frac{1}{6400}\left(120x^2 - x^3\right)$$

$$y' = \frac{1}{6400}\left(240x - 3x^2\right)$$

(3) differentiate

$$y'' = \frac{1}{6400}(240 - 6x)$$

(4) sign diagram



max slope $= y'(40) = \frac{3}{4}$

The run is $\frac{1}{3}$ and so the rise $= \frac{1}{4}$

(5) conclusion!

The max stair height is $\frac{1}{4}$ m.

Example 6: An 8-billion-bushel corn crop brings a price of \$2.40/bu. A commodity broker uses the rule of thumb: If the crop is reduced by *x* percent, then the price increases by 10*x* cents. Which crop size results in maximum revenue and what is the price per bushel?

(1) Table

	q	price	revenue.
	8	2.40	$8(2.4)$
-1%	7.92	2.50	$(7.92)(2.5)$
-2%	7.84	2.60	⋮
-3%	7.76	2.70	
⋮			
-x%	$8-.08x$	$2.40+.10x$	$R(x)=(8-.08x)(2.4+.10x)$

(2) set up eqt.

$$R(x)=(8-0.08x)(2.4+0.10x)$$

(3) deriv,

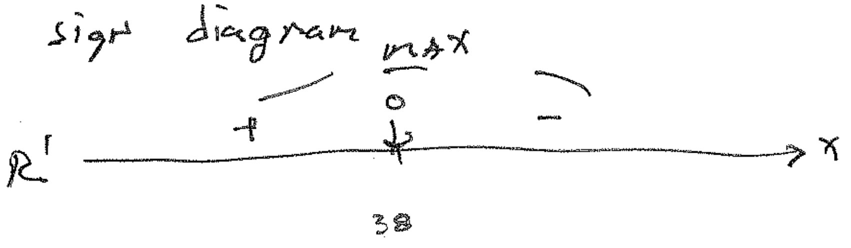
$$R'(x)=-0.08(2.4+.1x)+.1(8-0.08x)$$

$$=-.192-.008x+.8-.008x$$

$$=0.608-0.016x$$

$R'=0$ when $x=38$

(4) sign diagram

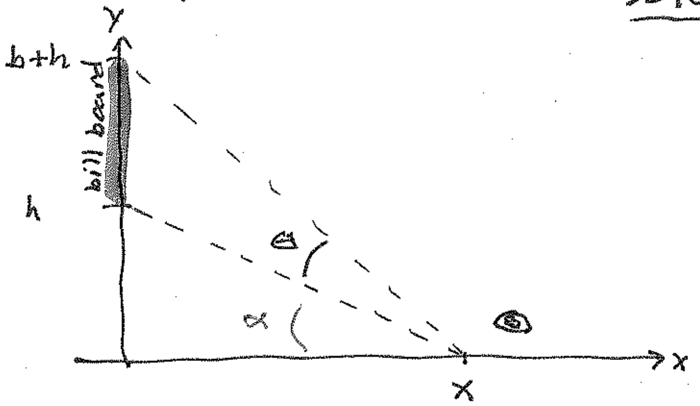


(5) our max rev. is when $8-38(.08)=4.96$ bil. bushels of corn are produced. Each bushel is sold for \$6.20/bu.

Example 7: A billboard of height *b* is mounted on the side of a building with its bottom edge at a distance *h* from the street below. At what distance *x* should an observer stand from the wall to maximize the angle of observation?

(1) Draw picture

solution 1: using arctan and detailed algebra



(2) set up eqt.

$\tan(\alpha) = \frac{h}{x}$ and $\tan(\alpha+\theta) = \frac{h+b}{x}$

$\Rightarrow \alpha = \arctan\left(\frac{h}{x}\right)$ and $\alpha+\theta = \arctan\left(\frac{h+b}{x}\right)$

$\Rightarrow \theta = \arctan\left(\frac{h+b}{x}\right) - \arctan\left(\frac{h}{x}\right)$

(3) differentiate

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{h+b}{x}\right)^2}\cdot\frac{-(h+b)}{x^2} - \frac{1}{1+\left(\frac{h}{x}\right)^2}\cdot\frac{-h}{x^2}$$

$$= -\frac{x^2}{x^2+(h+b)^2}\cdot\frac{(h+b)}{x^2} + \frac{x^2}{x^2+h^2}\cdot\frac{h}{x^2}$$

$$= \frac{h}{x^2+h^2} - \frac{h+b}{x^2+(h+b)^2}$$

$$= \frac{h(x^2+h^2+2hb+b^2)-(h+b)(x^2+h^2)}{(x^2+h^2)(x^2+(h+b)^2)}$$

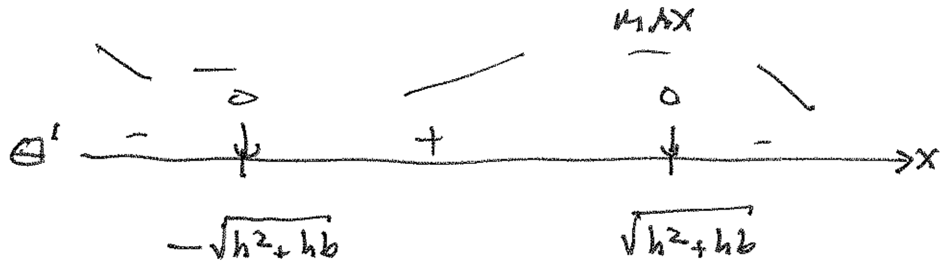
$$\begin{array}{r} hx^2+h^3+2h^2b+hb^2 \\ -hx^2-h^3-b\,h^2 \quad -bx^2 \\ \hline h^2b+hb^2-bx^2 \end{array}$$

which is $b(h^2+hb-x^2)$

$$= \frac{b((h^2+hb)-x^2)}{(x^2+h^2)(x^2+(h+b)^2)}$$

Note: the denominator >0 and the numerator has zeros @ $x = \pm\sqrt{h^2+hb}$

(4) sign diagram



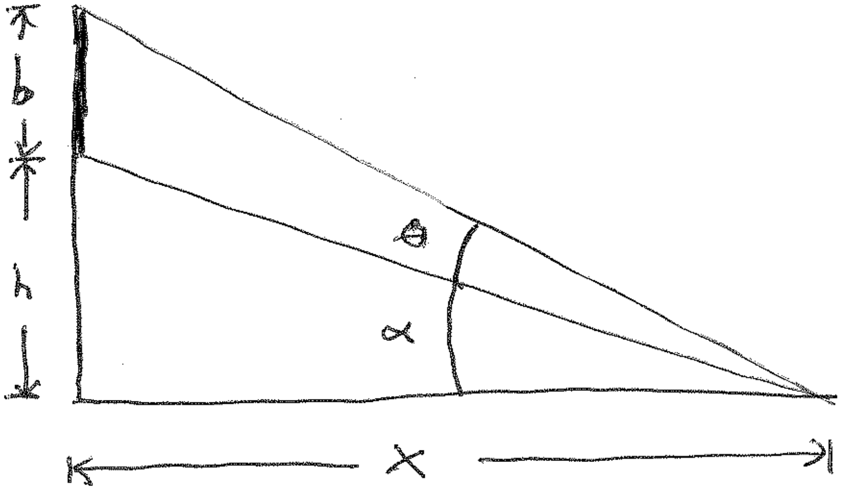
(5) conclusion

the viewing angle is optimized when the observer is a distance of $\sqrt{h^2+hb}$ from the building.

NOTE: This can also be done using the laws of cosines... but this approach is very painful.

Solution 2: using $\tan(\alpha+\theta)$ identity and easier algebra

(1) pic



(2) eqt.

$$\tan(\alpha+\theta) = \frac{b+h}{x}$$

$$\Rightarrow \frac{b+h}{x} = \frac{\tan\alpha + \tan\theta}{1-\tan\alpha\tan\theta}$$

$$\Rightarrow \frac{b+h}{x} = \frac{\frac{h}{x} + \tan\theta}{1 - \frac{h}{x}\cdot\tan\theta}$$

$$\Rightarrow \frac{b+h}{x}\left(1 - \frac{h}{x}\tan\theta\right) = \frac{h}{x} + \tan\theta$$

$$\Rightarrow \frac{1}{x}(b+\cancel{h}-\cancel{h}) = \tan\theta\left(1+\frac{h(b+h)}{x^2}\right)$$

$$\Rightarrow \tan\theta = \frac{\frac{b}{x}}{\frac{x^2+h^2+hb}{x^2}}$$

$$\Rightarrow \tan\theta = \frac{xb}{x^2+h^2+hb}$$

(3) differentiate implicitly.

$$\sec^2\theta \frac{d\theta}{dx} = \frac{b(x^2+h^2+hb) - 2x(xb)}{(x^2+h^2+hb)^2}$$

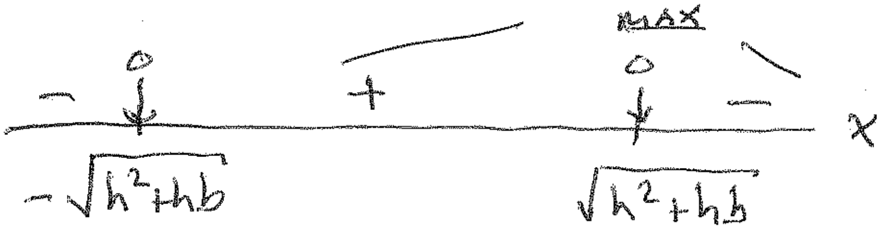
$$\Rightarrow \frac{d\theta}{dx} = \frac{\cos^2\theta\,(h^2+hb-x^2)b}{(X^2+h^2+hb)^2}$$

Notice $0 < \theta < \frac{\pi}{2}$ so $\cos^2\theta \neq 0$ and $(X^2+h^2+hb)^2 \neq 0$ ... both terms are positive.

(4) optimize

$\frac{d\theta}{dx} = 0$ when $h^2+hb-x^2=0$

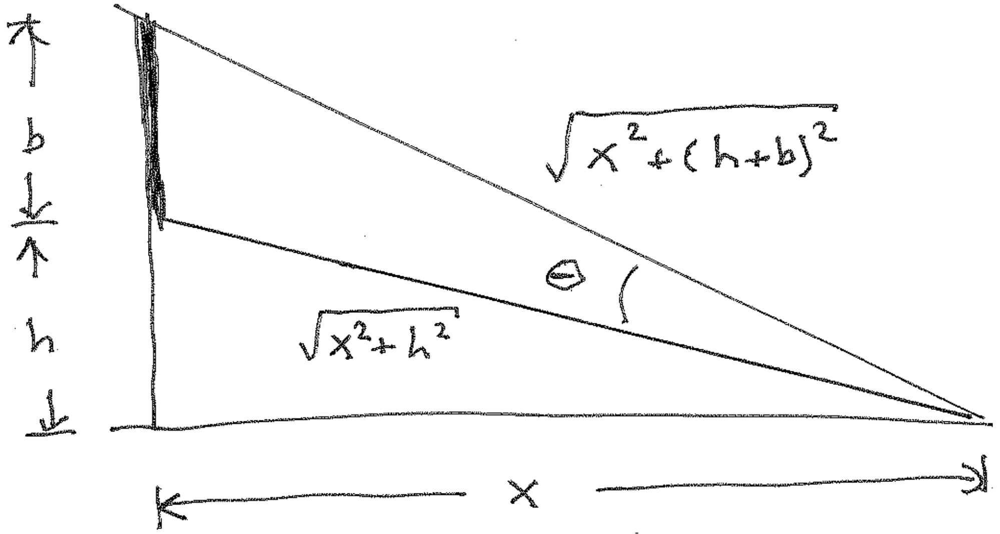
$$\Rightarrow x = \pm\sqrt{h^2+hb}$$



(5) Answer the Question.

So the best viewing angle is when you are $\sqrt{h^2+hb}$ from the building solution 3: using the law of cosines.

(1) pic



(2) eqc (by the law of cosines)

$$b^2 = (x^2+h^2)+(x^2+(h+b)^2) - 2\sqrt{x^2+h^2}\sqrt{x^2+(h+b)^2}\cos\theta$$

$$\Rightarrow \theta = \arccos\left(\frac{2x^2+2h^2+2hb}{2\sqrt{x^2+h^2}\sqrt{x^2+(h+b)^2}}\right)$$

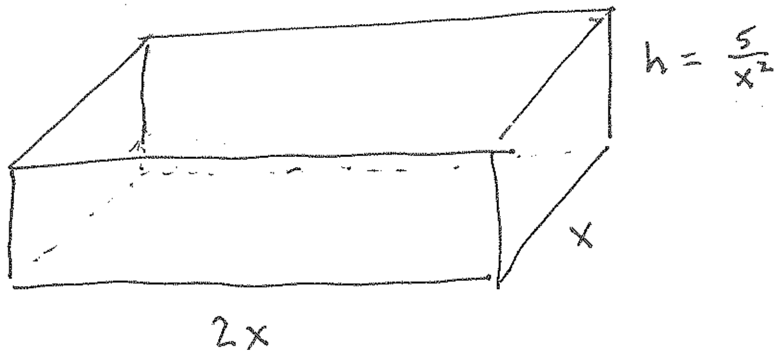
$$= \arccos\left(\frac{x^2+h^2+hb}{\sqrt{x^2+h^2}\sqrt{x^2+(h+b)^2}}\right)$$

Example 8: A rectangular storage container with an open top is to have a volume of 10 ft^3. The length of its base is twice the width. Material for the base costs \$3 per ft^2 while material for the sides costs \$4 per ft^2. Find the cost of materials for the cheapest such container.

$$10 = 2x^2 \cdot h$$

$$\Rightarrow h = \frac{5}{x^2}$$

(1) draw a picture



(2) set up eqts.

$$C(x) = \text{bottom} + \text{2 sides} + \text{front/back.}$$

$$= 3(2x \cdot x) + 2 \cdot 4\left(x \cdot \frac{5}{x^2}\right) + 2 \cdot 4\left(2x \cdot \frac{5}{x^2}\right)$$

$$= 6x^2 + \frac{120}{x}$$

(3) take derivative.

$$C'(x) = 12x - \frac{120}{x^2}$$

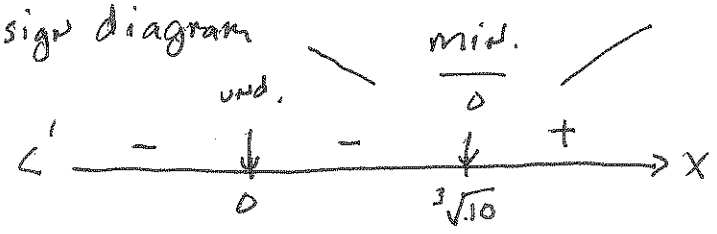
$$= \frac{12x^3 - 120}{x^2}$$

solve $12x^3 - 120 = 0$

$$\Rightarrow x^3 = \frac{120}{12} = 10$$

$$\Rightarrow x = \sqrt[3]{10}$$

(4) sign diagram



(5) answer the question.

$$C(\sqrt[3]{5}) = 6 \cdot 10^{2/3} + 60 \cdot 10^{-1/3} \approx 83.55$$

The min cost was \$83.55