Math 151
Curve Sketching Workalong

Example 1 to 4 are based on Khan Academy videos. As with all of Sal's films (Sal is the founder of Khan Academy), these are solid and thorough. However Sal doesn't use the sign diagram/chart approach that I like and so I am including that as well in the handout to help you.

Video 1: "Calculus: Graphing Using Derivatives" with URL https://youtu.be/hlgnece9ins

Example 1: (start 0:00 and end 20:30) Use calculus to sketch a graph of $f(x)=3x^4-4x^3+2$

$f(x)=3x^4-4x^3+2$

$f'(x)=$ $12x^3 - 12x^2$

$f''(x)=$ $36x^2 - 24x$

The zeros of $f''(x)$

solve $36x^2 - 24x = 0$

$\Rightarrow 12x(3x-2)=0$

$\Rightarrow x=0$ OR $x=\frac{2}{3}$

Critical points: Where $f'(x)=0$ or is undefined

solve $12x^3 - 12x^2 = 0$

$\Rightarrow 12x^2(x-1)=0$

$\Rightarrow x=0$ OR $x=1$

$f''(0)=0$ so the 2nd deriv. test is inconclusive.

$f''(1)=12>0$ so f is smiley and has a local min when $x=1$

Note: Because Sal doesn't use a sign diagram/chart, he makes use of the Second Derivative Test from section 4.3. This happens around 6:30 into the video. The Test says: Suppose f'' is continuous near *c*.

a.) If $f'(c)=0$ and $f''(c)>0$, then *f* has a local minimum at *c*.
b.) If $f'(c)=0$ and $f''(c)<0$, then *f* has a local maximum at *c*.
c.) If $f'(c)=0$ and $f''(c)=0$, then the second derivative test is inconclusive (it failed).

Begins at 8:30

What happens at $x=\frac{2}{3}$

$x>\frac{2}{3} \Rightarrow f''(x)>0$

$f''(x) = \underbrace{12x}_{+}\underbrace{(3x-2)}_{+}$

$x<\frac{2}{3} \Rightarrow f''(x)<0$

$f''(x) = \underbrace{12x}_{+}\underbrace{(3x-2)}_{-}$

Begins at 12:10

What happens at $x=0$

$x>0 \Rightarrow f''(x)<0$

$f''(x) = \underbrace{12x}_{+}\underbrace{(3x-2)}_{-}$

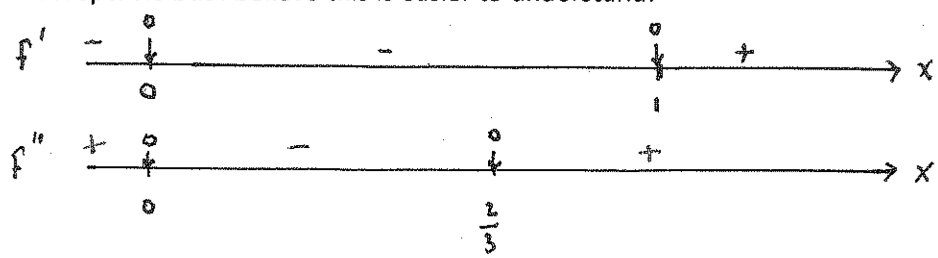
$x<0 \Rightarrow f''(x)>0$

$f''(x) = \underbrace{12x}_{-}\underbrace{(3x-2)}_{-}$

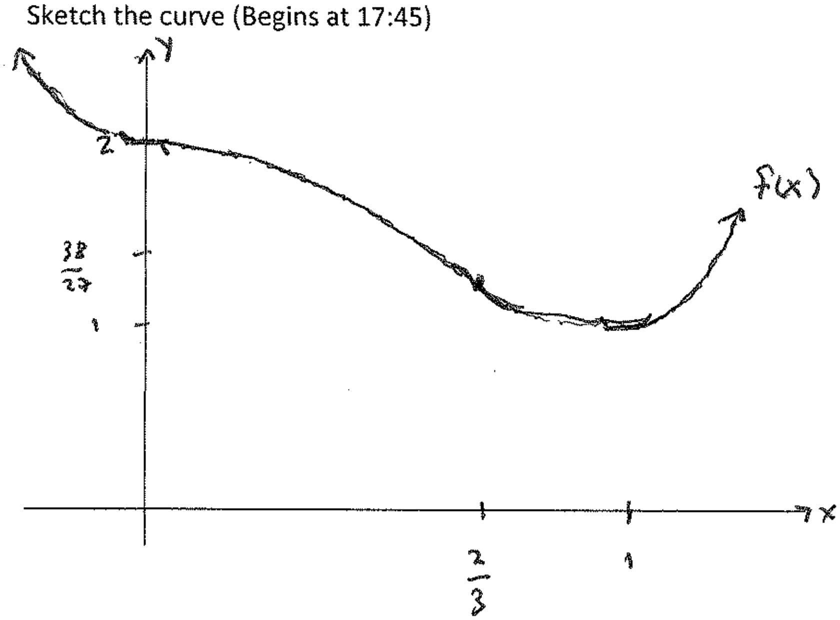
Summary (Begins at 14:45)

x value	Slope	What happens on the graph	*y* value
$x = 1$	slope = 0	concave up	$f(1) = 1$
$x = 0$	slope = 0	inflaction point $x < 0$ up $x > 0$ down	$f(0) = 2$
$x = \frac{2}{3}$		inflection point $x < \frac{2}{3}$ down $x > \frac{2}{3}$ up	$f(\frac{2}{3}) = \frac{38}{27}$

Sal doesn't use a sign diagram/chart, but here is how I would capture the same information. Both are acceptable but I believe this is easier to understand.



Sketch the curve (Begins at 17:45)



Video 2: "Calculus Graphing with Derivatives Example" with URL https://youtu.be/zC_dTaEY2AY

<u>Example 2</u>: (start 0:00 and end 25:07) Use calculus to sketch a graph of $f(x)=\ln(x^4+27)$

$$f(x)=\ln(x^4+27)$$

$$f'(x)=\frac{4x^3}{x^4+27}$$

Critical points (Time 6:15)

$x = 0$

and $f(0) = \ln 27 \approx 3.3$

$(0, 3.3)$

$$f''(x)=\frac{12x^2(x^4+27)-(4x^3)^2}{(x^4+27)^2}$$

$$=\frac{324x^2-4x^6}{(x^4+27)^2}$$

$$=\frac{4x^2(81-x^4)}{(x^4+27)^2}$$

$$=\frac{4x^2(9+x^2)(9-x^2)}{(x^4+27)^2}$$

$$=\frac{4x^2(9+x^2)(3-x)(3+x)}{(x^4+27)^2}$$

If x is an inflection point, then $f''(x)=0$ (Time 8:15)

$\Rightarrow 4x^2(9+x^2)(3-x)(3+x)=0$

$\Rightarrow x = 0, \underbrace{\pm 3}_{\text{inflection}}$

$(0, 3.3) \leftarrow$ min

$\left.\begin{matrix}(3, 4.7)\\(-3, 4.7)\end{matrix}\right\}$ inflection points

$x < 0 \Rightarrow f''(x) > 0$

$x > 0 \Rightarrow f''(x) > 0$

So $x = 0$ is not an inflaction point.

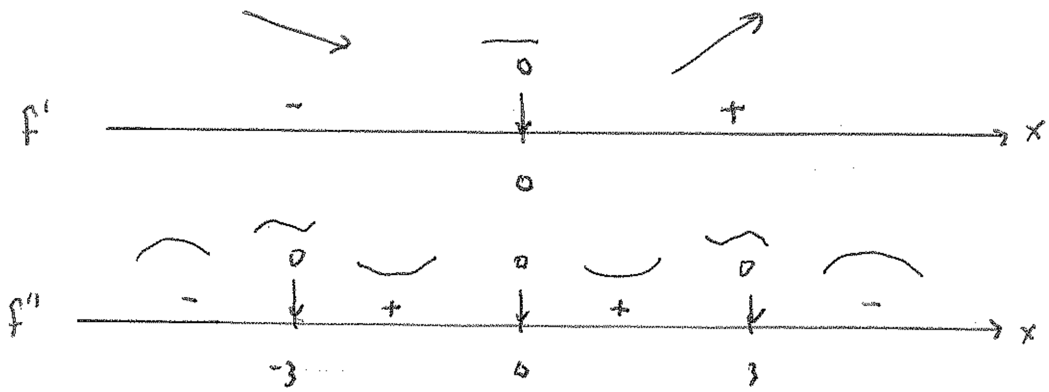
$x < 3 \Rightarrow f''(x) > 0$

$x > 3 \Rightarrow f''(x) < 0$

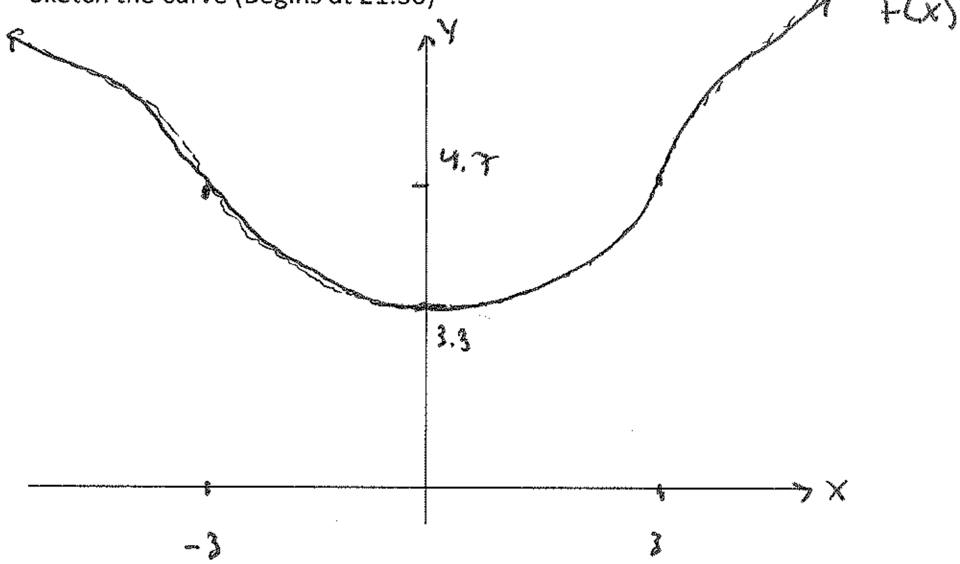
$x < -3 \Rightarrow f''(x) < 0$

$x > -3 \Rightarrow f''(x) > 0$

Again, Sal doesn't use a sign diagram/chart, but here is how I would capture the same information. Both are acceptable but I believe this is easier to understand.



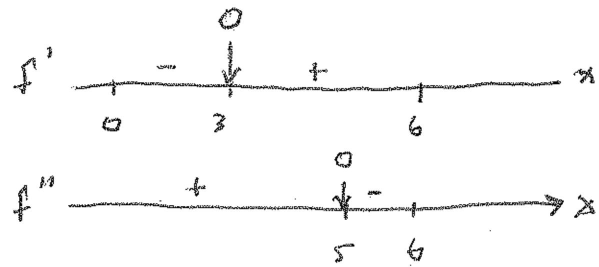
Sketch the curve (Begins at 21:30)



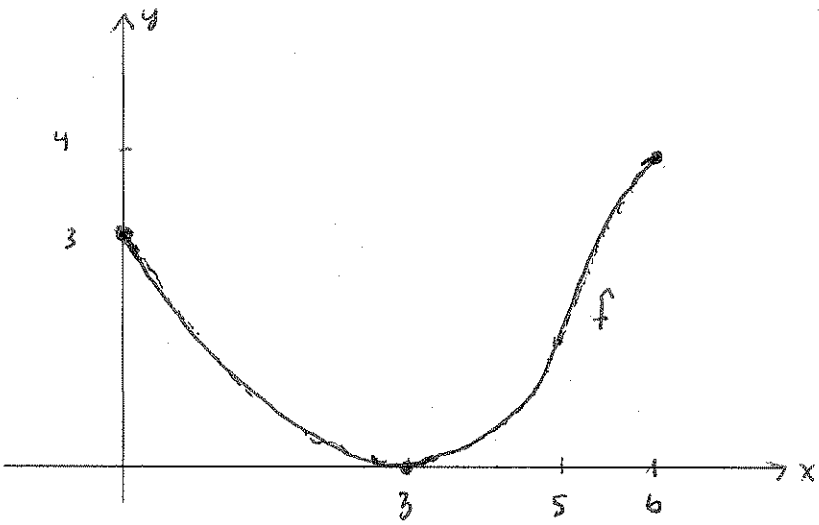
Video 3: "Graphing with Calculus" with URL https://youtu.be/ojcp0GJKluM

Example 3: (start 0:00 and end 8:18) Sketch the curve of f that has the following properties:

- $f(0)=3$, $f(3)=0$, and $f(6)=4$
- $f'(x)<0$ on (0,3)
- $f'(x)>0$ on (3,6)
- $f''(x)>0$ on (0,5)
- $f''(x)<0$ on (5,6)
- $f''(5)=0$



Sketch the curve (Begins at 6:00)



Example 4: (start 8:18 and end 4:43) Prove a quadratic has no point of inflection

General quadratic $f(x) = ax^2 + bx + c$, $a \neq 0$

$\Rightarrow f'(x) = 2ax + b$

$\Rightarrow f''(x) = 2a$

since $a \neq 0$ we know $f'' \neq 0$ so there can't be an inflection point.

Example 5: Sketch the curve of $y = \frac{x-1}{x^2}$ using the methods of calculus. The solution to this example is spread across two videos (the total length is about 20 minutes).

Video 4: "Curve Sketching Using Calculus - Part 1 of 2" with URL https://youtu.be/vOTTuZflAIM
Video 5: "Curve Sketching Using Calculus - Part 2 of 2" with URL https://youtu.be/0Fx6jec8SwY

A Summary of curve sketching
Find
a.) Domain (Video 4, time 0:40)
b.) Intercepts (Video 4, time 1:20)
c.) Symmetry (Video 4, time 2:30)
d.) Asymptotes (Video 4, time 4:20)
e.) Intervals of increase/decrease (Video 4, time 6:20)
f.) Local max/mins (Video 4, time 9:20)
g.) Concavity and points of inflection (Video 5, time 0:00)
h.) Sketch (Video 5, time 4:10)

a) Domain.

$(-\infty, 0) \cup (0, \infty)$

b) Intercepts

x-intercept: solve $x - 1 = 0$

$\Rightarrow x = 1$

y-intercept: none.

c) Symmetry

$$y(-x) = \frac{-x-1}{(-x)^2}$$

$$= \frac{-(x+1)}{x^2}$$

No symmetry.

d) asymptotes

V.A. $\quad x = 0$

H.A. $\quad \displaystyle\lim_{x \to \pm\infty} \frac{x-1}{x^2} = 0$

so a H.A. @ $y = 0$

e) intervals of increasing/decreasing.

$$y' = \frac{1 \cdot x^2 - 2x(x-1)}{(x^2)^2}$$

$$= \frac{2x - x^2}{x^4}$$

$$= \frac{2-x}{x^3}$$

und. $\qquad$ 0

y' $\quad - \quad | \quad + \quad | \quad - \quad \to x$

$\qquad\quad 0 \qquad\quad 2$

increasing: $(0, 2)$

decreasing: $(-\infty, 0) \cup (2, \infty)$

f) local max/min

There is a local max @ $(2, \frac{1}{4}$

$$y(2) = \frac{2-1}{4} = \frac{1}{4}$$

g) concavity

$$y'' = \frac{-1 \cdot x^3 - 3x^2(2-x)}{(x^3)^2}$$

$$= \frac{2x^3 - 6x^2}{x^6}$$

$$= \frac{2x-6}{x^4}$$

und. $\qquad$ 0

y'' $\quad - \quad | \quad - \quad | \quad + \quad \to x$

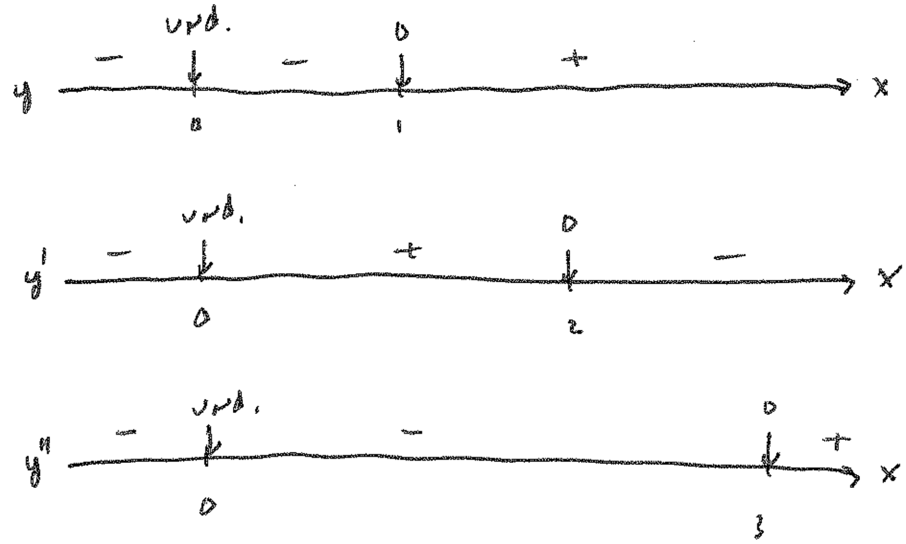
$\qquad\quad 0 \qquad\quad 3$

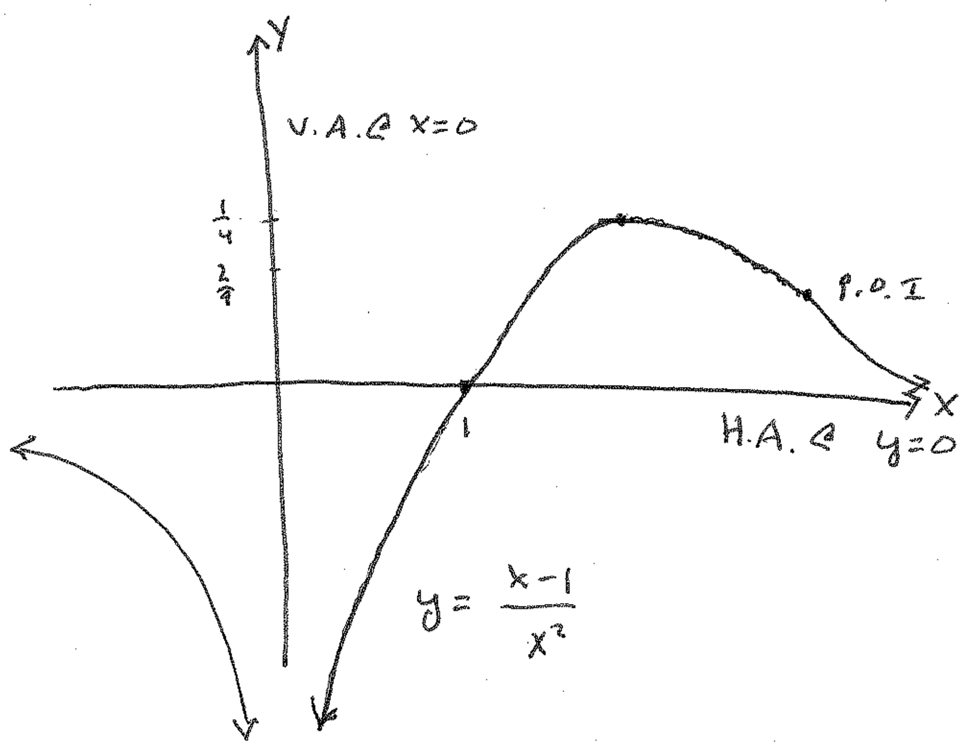
concave up: $(3, \infty)$

concave down: $(-\infty, 0) \cup (0, 3)$

point of inflection: $(3, \frac{2}{9})$

h) sketch





Example 6: Sketch the curve of $f(x)=\dfrac{x}{\sqrt{x^2+1}}$ using the methods of calculus. The solution to this example is spread across four videos (the total length is about 35 minutes).

Video 6: "Summary of Curve Sketching - Example 2, Part 1 of 4" at https://youtu.be/DMYUsv8ZaoY
Video 7: "Summary of Curve Sketching - Example 2 - Part 2 of 4" at https://youtu.be/HHeYsgNzKeE
Video 8: "Summary of Curve Sketching - Example 2 - Part 3 of 4" at https://youtu.be/oy-x-xGWAf4
Video 9: "Summary of Curve Sketching - Example 2 - Part 4 of 4" at https://youtu.be/DO2NHtTG0TM

A Summary of curve sketching
Find
- a.) Domain (Video 6, time 0:40)
- b.) Intercepts (Video 6, time 3:20)
- c.) Symmetry (Video 6, time 5:30)
- d.) Asymptotes (Video 7, time 0:00)
- e.) Intervals of increase/decrease (Video 8, time 0:00)
- f.) Local max/mins (Video 8, time 8:50)
- g.) Concavity and points of inflection (Video 8, time 9:00)
- h.) Sketch (Video 9, time 0:00)

a) domain

$(-\infty, \infty)$

b) intercepts

$f(x)=0$ when $x=0$

c) symmetry

$$f(-x)=\frac{-x}{\sqrt{(-x)^2+1}}$$

$$=\frac{-x}{\sqrt{x^2+1}}$$

$$=-f(x)$$

f is odd - symmetric about the origin.

d) asymptotes

V.A. – none

H.A.: $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{x^2+1}} = 1$

$\lim\limits_{x \to -\infty} \dfrac{x}{\sqrt{x^2+1}} = -1$

e) intervals of increase/decrease.

$$f'(x) = \frac{1\sqrt{x^2+1} - x \cdot \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{2(x^2+1) - x^2}{(x^2+1)^{3/2}}$$

$$= \frac{x^2+2}{(x^2+1)^{3/2}}$$

$+$

x

increasing $(-\infty, \infty)$

f) local max/min.: none

g) concavity

$$f''(x) = \frac{2x(x^2+1)^{3/2} - (x^2+2)\cdot\frac{3}{\cancel{2}}(x^2+1)^{1/2}\cdot\cancel{2}x}{\left[(x^2+1)^{3/2}\right]^2}$$

$$= \frac{2x(x^2+1) - 3x(x^2+2)}{(x^2+1)^{5/2}}$$

$$= \frac{-x^3-4x}{(x^2+1)^{5/2}}$$

$$= \frac{-x(x^2+4)}{(x^2+1)^{5/2}}$$

point of inflection @ (0,0)

f'': $+$ on $x<0$, 0 at $x=0$, $-$ on $x>0$

h) sketch