2.3: limit laws (or 2.6)

$$\lim_{x \to \infty} \frac{-2x^{12} - 2x + 3}{5x^{13} + 3x - 7} \to \frac{-\infty}{\infty}$$

soln 1

$$= \lim_{x \to \infty} \frac{-2 + \frac{-2}{x^{11}} + \frac{3}{x^{12}}}{5x + \frac{3}{x^{11}} - \frac{7}{x^{12}}}$$

divide numerator and denom. by x^2.

$$= \frac{-2}{\infty}$$

$$= 0$$

soln 2

$$\lim_{x \to \infty} \frac{-2x^2 - 2x + 3}{5x^3 + 3x - 7} \to \frac{-\infty}{\infty}$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{-4x - 2}{15x^2 + 3} \to -\frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{-4}{30x + 0} \to \frac{-4}{\infty}$$

$$= 0$$

2.4: Precise Dfn. of the Limit

<u>Claim!</u> $\lim_{x \to 2} (3x - 17) = -11$

<u>proof.</u>

Let $\varepsilon > 0$ be given

choose $\delta = \frac{\varepsilon}{3}$

If $0 < |x-2| < \delta$

$\Rightarrow |x-2| < \frac{\varepsilon}{3}$

$\Rightarrow 3|x-2| < \varepsilon$

$\Rightarrow |3x - 6| < \varepsilon$

$\Rightarrow |(3x-17)+11| < \varepsilon$

Hence $\lim_{x \to 2} (3x-17) = -11$.

Scratch work

$|f(x) - \overset{(-11)}{\cancel{L}}| < \varepsilon$

$\Rightarrow |(3x-17)+11| < \varepsilon$

$\Rightarrow |3x-6| < \varepsilon$

$\Rightarrow 3|x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{3}$

$\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$

there exists $\delta > 0$ s.t. if $0 < |x-a| < \delta$

then $|f(x) - L| < \varepsilon$.

$\boxed{\text{Precise Dfn. of the Limit}}$

2.4: Precise Dfn of the Limit

Claim: $\lim_{x \to 1} (2x+3) = 5$

proof.

Let $\varepsilon > 0$ be given

Choose $\delta = \frac{\varepsilon}{2}$

If $0 < |x-1| < \delta$

$\Rightarrow |x-1| < \frac{\varepsilon}{2}$

$\Rightarrow 2|x-1| < \varepsilon$

$\Rightarrow |2x-2| < \varepsilon$

$\Rightarrow |(2x+3)-5| < \varepsilon$

Hence $\lim_{x \to 1} (2x+3) = 5$.

Scratch

$|f(x) - 5| < \varepsilon$

$\Rightarrow |2x+3-5| < \varepsilon$

$\Rightarrow |2x-2| < \varepsilon$

$\Rightarrow 2|x-1| < \varepsilon$

$\Rightarrow |x-1| < \frac{\varepsilon}{2}$

2.5: squeeze thm

$$\lim_{x \to 0} \left(1 + x^2 \cos\left(\frac{1}{x}\right)\right)$$

* squeeze thm

If $f(x) \le g(x) \le h(x)$ near $x = a$ (except possibly at $x = a$) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

Then $\lim_{x \to a} g(x) = L$.

* find the squeezing fcns

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

$$\Rightarrow -x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

needed.

$$\Rightarrow 1 - x^2 \le 1 + x^2 \cos\left(\frac{1}{x}\right) \le 1 + x^2$$

and $\lim_{x \to 0} (1 - x^2) = 1 = \lim_{x \to 0} (1 + x^2)$

hence $\lim_{x \to 0} \left(1 + x^2 \cos\left(\frac{1}{x}\right)\right) = 1$ by the squeeze thm.

3.5: Implicit Differentiation.

Find $\frac{dy}{dx}$ if $e^y \sin x = x + xy$

$$\Rightarrow \frac{d}{dx}\left(e^y \sin x\right) = \frac{d}{dx}(x + xy)$$

$$\Rightarrow \underline{e^y \cdot y' \sin x} + \cos x \, e^y = 1 + y + \underline{x y'}$$

$$\Rightarrow y' e^y \sin x - x y' = 1 + y - \cos x \, e^y$$

$$\Rightarrow y' = \frac{1 + y - \cos x \, e^y}{e^y \sin x - x}$$

3.9: Linear Approx

claim in physics that $\sin x \approx x$ for x near 0.

Let $f(x) = \sin x \Big|_{x=0} \; 0$

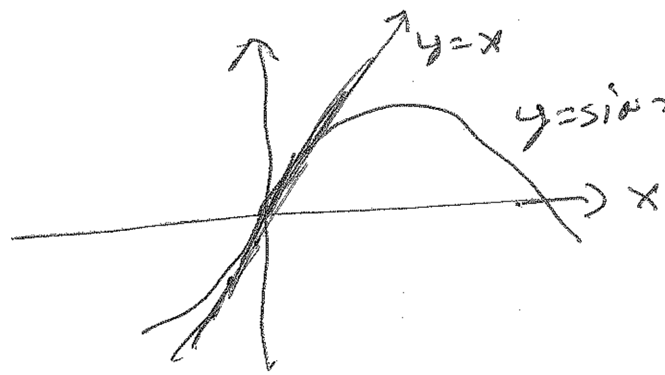
$f'(x) = \cos x \Big|_{x=0} \; 1$

slope $= 1$; pt $(0,0)$

Tangent/lin. approx is $y - 0 = 1(x - 0)$

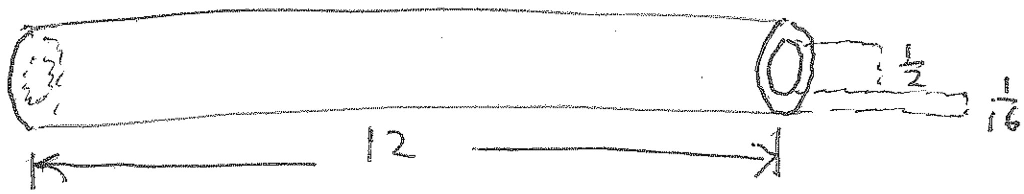
or $y = x$.

Pic



3.10: Differentials

Use differentials to est. the material needed in a steel pipe w/ inside diameter $\frac{1}{2}$" and wall thickness $\frac{1}{16}$" and length 12".



Volume of cylinder

$$V = \pi r^2 \cdot L$$

$$\Rightarrow dV = 2\pi r \, dr \, L \quad \left(\text{found } \frac{dV}{dr}\right)$$

$$r = .25 \; ; \; dr = 0.0625, \; L = 12$$

$$\Rightarrow dV = 1.17 \text{ in}^3$$

the amount of steel needed is about 1.17 in^3.

4.4: l'Hospital's Rule

$$\lim_{x\to 0} (\csc x - \cot x)$$

$$= \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) \to \infty - \infty$$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \to \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\sin x}{\cos x} \to \frac{0}{1}$$

$$= 0$$

4.4: L'Hospital's Rule

$$\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right)$$

$$= \lim_{x \to 0} \left(\frac{\cos x}{\sin x} - \frac{1}{x} \right) \to \infty - \infty$$

$$= \lim_{x \to 0} \frac{x \cos x - \sin x}{x \sin x} \to \frac{0}{0}$$

H

$$= \lim_{x \to 0} \frac{\cancel{\cos x} - x \sin x - \cancel{\cos x}}{\sin x + x \cos x} \to \frac{0}{0}$$

H

$$= \lim_{x \to 0} \frac{-(\sin x + x \cos x)}{\cos x + \cos x - x \sin x} \to \frac{0}{2}$$

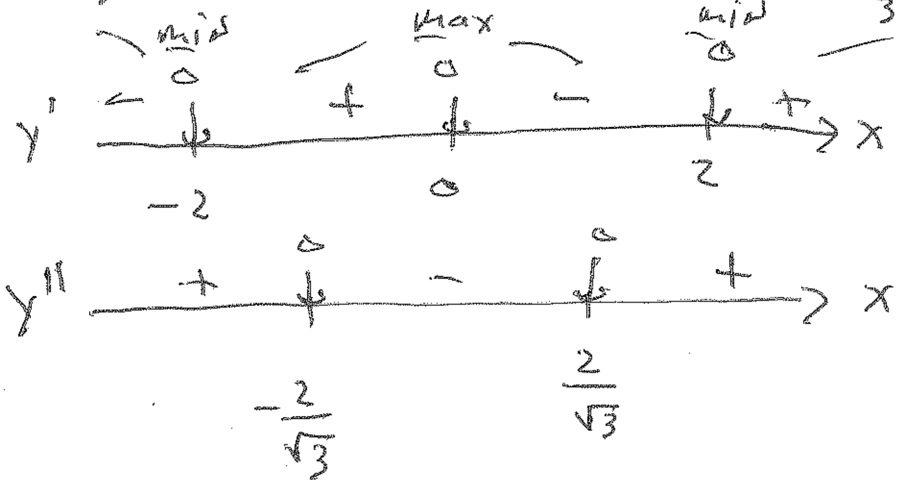
$$= 0$$

4.5: curve sketching

sketch $y = x^4 - 8x^2 + 8$

$$y' = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2)$$

$$y'' = 12x^2 - 16 = 12\left(x^2 - \frac{4}{3}\right) = 12\left(x + \frac{2}{\sqrt{3}}\right)\left(x - \frac{2}{\sqrt{3}}\right)$$



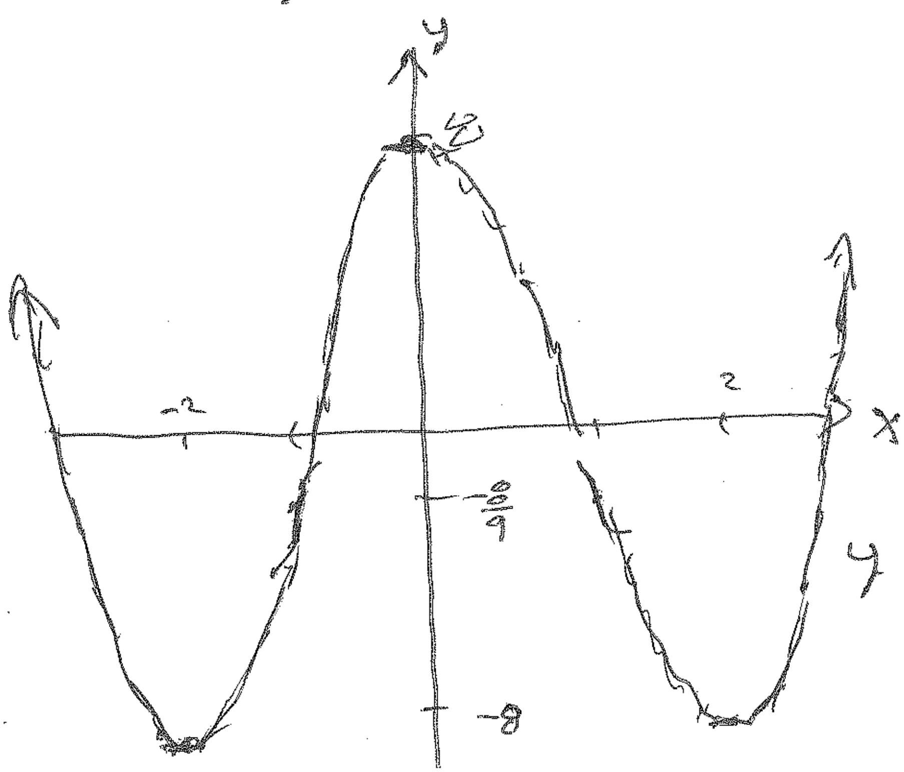
min $(-2, -8)$

$(2, -8)$

max $(0, 8)$

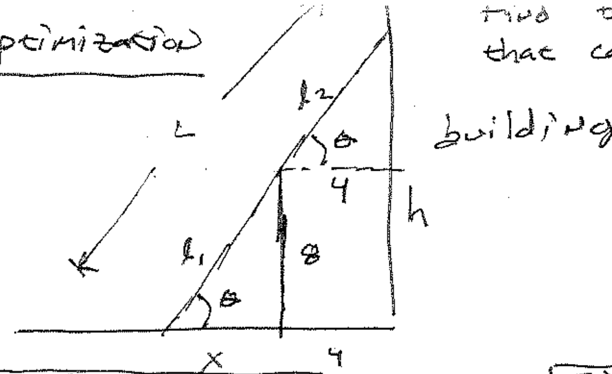
poi $\left(-\frac{2}{\sqrt{3}}, -\frac{8}{9}\right)$

$\left(\frac{2}{\sqrt{3}}, -\frac{8}{9}\right)$



$y = x^4 - 8x^2 + 8$

4.7: optimization



Find the shortest ladder that can reach from the ground to a building over an 8ft fence 4 ft from the bldg

$\boxed{\text{similar Triangles}}$

$$\frac{x}{8} = \frac{x+4}{h}$$

$$\Rightarrow h = \frac{8(x+4)}{x}$$

$$\text{and } L = \sqrt{(x+4)^2 + h^2}$$

$$= \sqrt{(x+4)^2 + \left[\frac{8(x+4)}{x}\right]^2}$$

$$= \sqrt{\frac{x^2(x+4)^2 + 64(x+4)^2}{x^2}}$$

$$= \frac{(x+4)}{x}\sqrt{x^2+64}$$

$$\Rightarrow L' = \frac{x^3 - 256}{x^2\sqrt{x^2+64}}$$

for $x > 0$, the only real critical number is at $x = \sqrt[3]{256}$

min

$$L' \quad \overset{-}{} \quad \underset{\sqrt[3]{256}}{\overset{0}{\downarrow}} \quad \overset{+}{} \longrightarrow x$$

so the min is @ $x = \sqrt[3]{256}$ and $L(\sqrt[3]{256}) = 16.6478$

so the shortest ladder is 16.65ft long.

$\boxed{\text{Trig}}$

$$L = l_1 + l_2$$

$$= 8\csc\theta + 4\sec\theta$$

$$\Rightarrow L' = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta$$

$$= \frac{4\sin\theta}{\cos^2\theta} - \frac{8\cos\theta}{\sin^2\theta}$$

and solve for $L' = 0$

$$\Rightarrow \frac{4\sin\theta}{\cos^2\theta} = \frac{8\cos\theta}{\sin^2\theta}$$

$$\Rightarrow 4\sin^3\theta = 8\cos^3\theta$$

$$\Rightarrow \tan^3\theta = 2$$

$$\Rightarrow \tan\theta = \sqrt[3]{2}$$

$$\Rightarrow \theta = \arctan(\sqrt[3]{2}) \approx 0.899908$$

min

$$L' \quad \overset{-}{} \quad \underset{0.899908}{\overset{0}{\downarrow}} \quad \overset{+}{} \longrightarrow \theta$$

and $L(0.899908) = 16.647$

so again the shortest ladder is 16.65 ft long.

4.8: Newton's Method

Note: $\cosh(x)$ = hyperbolic cosine

$$= \frac{e^{x} + e^{-x}}{2}$$

ex: Newton's method.

approx $\sqrt[5]{17}$

Let $\alpha = \sqrt[5]{17}$

$\Rightarrow \alpha^5 = 17$

$\Rightarrow \alpha^5 - 17 = 0$

Let $\alpha_1 = 1.5$

$$\alpha_n = \alpha_{n-1} - \frac{f(\alpha_{n-1})}{f'(\alpha_{n-1})}$$

where $f(\alpha) = \alpha^5 - 17$

$\alpha = 1.7623$

4.9: Antiderivatives

Suppose the acceleration of a particle is given by $a(t) = t^2 - 4t + 6$, when $t=0$ the position is 5 and when $t=1$, the position was 20. Find ~~the~~ pos. fct.

$$a(t) = t^2 - 4t + 6$$

velocity

$$v(t) = \frac{t^3}{3} - 2t^2 + 6t + C$$

position

$$s(t) = \frac{t^4}{12} - \frac{2}{3}t^3 + 3t^2 + Ct + D$$

and $s(0) = 5$, $s(1) = 20$

solve for C, D.

$$s(0) = 5 = 0 - 0 + 0 + 0 + D$$

$$\Rightarrow D = 5$$

$$s(1) = 20 = \frac{1}{12} - \frac{2}{3} + 3 + C + 5$$

$$\Rightarrow C = \frac{151}{12}$$

hence $s(t) = \frac{t^4}{12} - \frac{2}{3}t^3 + 3t^2 + \frac{151}{12}t + 5$

4.9: Antiderivatives

$$g(t) = \frac{3 + t + t^2}{\sqrt{t}}$$

$$= \frac{3}{\sqrt{t}} + \sqrt{t} + t^{3/2}$$

$$= 3t^{-1/2} + t^{1/2} + t^{3/2}$$

$$\Rightarrow G(t) = \frac{3t^{1/2}}{1/2} + \frac{t^{3/2}}{3/2} + \frac{t^{5/2}}{5/2} + C$$

$$= 6t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$$

4.9: Antiderivatives

If $f'(t) = \frac{3t^4 - t^3 + 6t^2}{t^4}$, find $f(t)$.

$$= \frac{3t^4}{t^4} - \frac{t^3}{t^4} + \frac{6t^2}{t^4}$$

$$= 3 - \frac{1}{t} + \frac{6}{t^2}$$

$$= 3 - \frac{1}{t} + 6t^{-2}$$

$$\Rightarrow \quad f(t) = 3t - \ln|t| - 6t^{-1} + C$$

Notational Note

"$=$" vs "$\Rightarrow$"

"$=$" joins expressions

"$\Rightarrow$" joins eqns/ inequalities.

ex $4x-1=7$

$\Rightarrow 4x=8$

ex $y = \frac{2x}{2}$

$= x$