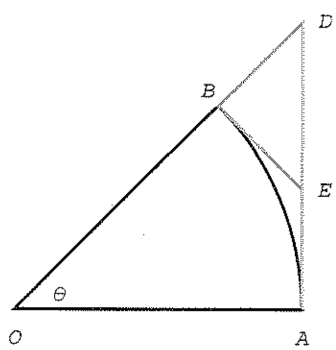
Proving that $\frac{d}{dx}\sin(x) = \cos(x)$.

To prove that $\frac{d}{dx}\sin(x) = \cos(x)$, we first prove that $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$ and $\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0$.

Claim A: $\frac{d}{dx}\sin(x) = \cos(x)$.

Recall that $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$.



Claim B: $\lim_{\theta \to 0} \frac{\cos(\theta)-1}{\theta} = 0$.

To find, $\lim_{\theta \to 0} \frac{\cos(\theta)-1}{\theta}$, multiply the expression by the "conjugate" of the numerator.

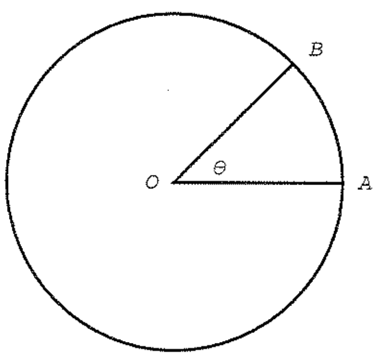
Claim C: $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$

To prove Claim A, we will use i.) trigonemetric geometry, ii.) the Squeeze Theorem, and iii.) we will call upon the symmetry of $\frac{\sin(\theta)}{\theta}$. Since we will use the Squeeze Theorem, we need an upper and lower bound for $\frac{\sin\theta}{\theta}$ near $\theta = 0$.

i.) Trigonometric Geometry

Finding an upper bound to $\frac{\sin(\theta)}{\theta}$.

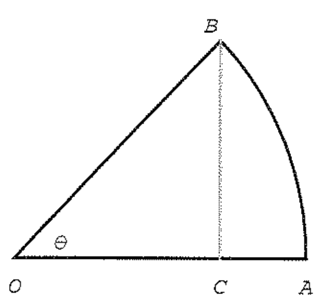
Consider the unit circle - specifically the sector of the circle with center O, central angle $0 < \theta < \frac{\pi}{2}$, and radius 1.



What are the coordinates of the point *B:*

How long is Arclength(AB):

Zooming in on the sector of the circle:

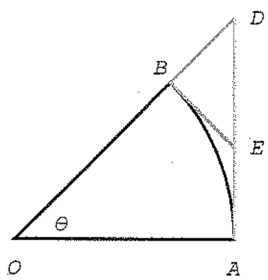


What is length(BC):

And since $\overline{\text{length (BC)}} < \overline{\text{arclength (AB)}}$, we have that $\sin(\theta) < \theta$ and hence $\frac{\sin(\theta)}{\theta} < 1$.

Finding a lower bound to $\frac{\sin(\theta)}{\theta}$.

Let the tangents at A and B intersect at point E.



Now, consider the following lengths in relation to eachother and to θ.

length(EB) $\qquad$ $\overline{\text{length (ED)}}$

length(ED) $\qquad$ length(AE)

length(AD) $\qquad$ length(AE) + length(ED)

$\theta =$ arclength(AB) $<$ length(AE) + length(EB) = ________________

Key point: ____

$\theta < \tan(\theta) \rightarrow \theta < \frac{\sin(\theta)}{\cos(\theta)} \rightarrow \cos(\theta) < \frac{\sin(\theta)}{\theta}$.

ii.) The Squeeze Theorem

In part i.) we showed that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$, for $0 < \theta < \frac{\pi}{2}$. Since $\lim_{\theta\to 0}\cos(\theta) = 1$and $\lim_{\theta\to 0} 1 = 1$, by the Squeeze Theorem we have that $\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$. Note: We only consider $\theta > 0$ since we are focused on the first quadrant.

iii.) Symmetry Argument.

If $f(\theta) = \frac{\sin(\theta)}{\theta}$, we have that $f(-\theta) = \frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin(\theta)}{\theta} = f(\theta)$. Hence, $f(\theta) = \frac{\sin(\theta)}{\theta}$is an even function. This means the left and right limits must be equal and so $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.