16.5: Curl & Divergence.

Overview

This section along w/ the next two are bridging sections.

The key concepts we will learn help us later. Curl $\Rightarrow$ Stoke's Thm.

Div $\Rightarrow$ Divergence Thm.

At the core, each of these Thms is similar to Green's Thm in that An integral over a boundary is equated to a multiple integral over the region w/in the boundary.

Curl & Div

The Del Operator. $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \quad (f \text{ a scalar fct})$$

Ex 0! If $f(x,y,z) = x^2 + y^3 + z^4$, find ∇f.

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad (\text{a scalar field})$$

we call this scalar product the <u>divergence of $\vec{F}$</u>

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}$$

<u>ex 1:</u> If $\vec{F} = e^{xy} \sin z \, \vec{j} + y \arctan\left(\frac{x}{z}\right) \vec{k}$
find $\operatorname{div}(\vec{F})$

$$\operatorname{div}(\vec{F}) = 0 + x e^{xy} \sin z + \frac{y}{1+\left(\frac{x}{z}\right)^2} \cdot \frac{-x}{z^2}$$

Now consider $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ (vector field)

we call this vector product the <u>curl of $\vec{F}$</u>

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$$

<u>ex 2!</u> (see ex 1). Find $\operatorname{curl}(\vec{F})$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & e^{xy} \sin z & y \arctan\left(\frac{x}{z}\right) \end{vmatrix}$$

$$= \left(\arctan\left(\frac{x}{y}\right) - e^{xy} \cos z\right)\vec{i} - \frac{y}{1+\left(\frac{x}{z}\right)^2} \cdot \frac{1}{z} \vec{j}$$

$$+ y e^{xy} \sin z \, \vec{k}$$

Div & Curl thms

16.5
3/6

Ex 3: (see ex c) If $f(x,y,z) = x^2 + y^3 + z^4$

find $\text{curl}(\vec{\nabla} f)$.

$$\text{curl}(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 3y^2 & 4z^3 \end{vmatrix}$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$$

Thm: If f is a fct of three variables that has cont. second order partials, then

$\text{curl}(\vec{\nabla} f) = \vec{0}$

□ proof (uses Clairaut's Thm)

⇒ If $\vec{F}$ is conservative, then $\text{curl}(\vec{F}) = \vec{0}$

Q: If $\text{curl}(\vec{F}) = \vec{0}$, does this mean that $\vec{F}$ is conservative?

Q: If $\text{curl}(\vec{F}) \neq \vec{0}$, does this mean $\vec{F}$ is not conservative?

Thm: If $\vec{F}$ is a vector field on all of $\mathbb{R}^3$ whose components have continuous partials and $\text{curl}(\vec{F}) = \vec{0}$, then $\vec{F}$ is a conservative vector field.

Ex5: Determine if $\vec{F} = \langle y\cos xy, x\cos(xy), -\sin z\rangle$ is conservative... if so... Find f.

Thm: If $\vec{F} = \langle P, Q, R\rangle$ is a vector field on $\mathbb{R}^3$ w/ cont. 2nd order partials of $P, Q, \& R$, then $\text{div}(\text{curl}(\vec{F})) = 0$

see analogy to $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$.

doesn't satisfy assumptions

Ex6: Show that the vector field $\vec{F} = \left\langle 0, e^{xy}\sin z, y\tan^{-1}\left(\frac{x}{z}\right)\right\rangle$ can't be written as the curl of another vector field, that is, $\vec{F} \neq \text{curl}\,\vec{G}$

see (ex1) ... $\text{div}(\vec{F}) \neq \vec{0} \Rightarrow \nexists\, \vec{G}$ s.t. $\vec{F} \neq \text{curl}(\vec{G})$.

$\boxed{\text{The Laplace Operator}}$

$$\text{div}(\vec{\nabla} f) = \nabla \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We call $\nabla^2 = \nabla \cdot \nabla$ the Laplace Operator which is tied to Laplace's Eqt.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

AND $$\nabla^2 \vec{F} = \nabla^2 P \vec{i} + \nabla^2 Q \vec{j} + \nabla^2 R \vec{k}$$

Div, curl, & Green's Thm

recall $$\oint_C F \cdot dr = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$

force field dotted w/ a differential.

scalar fct

(I) Curl If $\vec{F} = \langle P, Q, 0 \rangle$, then $\text{curl}(F) = \nabla \times \vec{F}$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$$

(no "z" in P & Q)

to get the $\vec{k}$ component out as a scalar fct...

$$\text{curl}(\vec{F}) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

AND $$\oint_C \vec{F} \cdot dr = \iint_D \text{curl}(\vec{F}) \cdot \vec{k}\, dA$$

The line integral of the tangential component of $\vec{F}$ along C is the double integral of the vertical component of $\text{curl}(\vec{F})$ over the region D enclosed by C.

Divergence

(II) If C is given by the vector equation

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \quad a \le t \le b$$

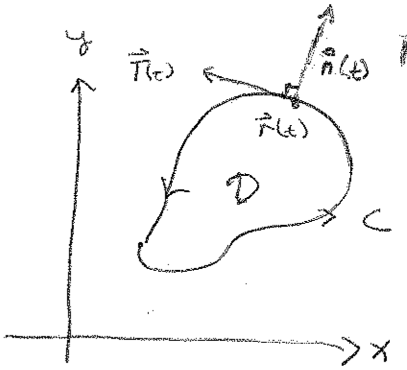
$$\Rightarrow \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$$ (unit tangent vector)

Notice that $\vec{T} \cdot \vec{n} = 0$.

Now define $\vec{n}(t)$ as

$$\vec{n}(t) = \frac{y'(t)}{|\vec{r}'(t)|}\vec{i} - \frac{x'(t)}{|\vec{r}'(t)|}\vec{j}$$

which is a normal vector to C pointing away from D.



$\vec{n}$ c.w. $90°$ from $\vec{T}$

recall $\int_C F(x,y)\,ds = \int_a^b F(\vec{r}(t))|\vec{r}'(t)|\,dt$

$F(x,y)$: scalar fct. ↓ $\vec{F} \cdot \vec{n}$ is a scalar fct.

$|\vec{r}'(t)|$: arclength is positive

Flux = So $\oint_C \vec{F} \cdot \vec{n}\,ds = \int_a^b (F \cdot n)(t)|\vec{r}'(t)|\,dt$

Flux: for a velocity field $\vec{F}$, the flux measures how much fluid passes thru C per unit time.

$$= \int_a^b \left[\frac{P(x(t),y(t))\,y'(t)}{\cancel{|\vec{r}'(t)|}} - \frac{Q(x(t),y(t))\,x'(t)}{\cancel{|\vec{r}'(t)|}}\right]\cancel{|\vec{r}'(t)|}\,dt$$

$$= \oint_C P\,dy - Q\,dx$$

Green's Thm.

$$= \iint_D \underbrace{\left[\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\right]}_{div(\vec{F})} dA$$

AND $\oint_C \vec{F} \cdot \vec{n}\,ds = \iint_D div(\vec{F})\,dA$ which means

the line integral of the normal component of $\vec{F}$ along C is equal to the double integral of the divergence of $\vec{F}$ over the region D