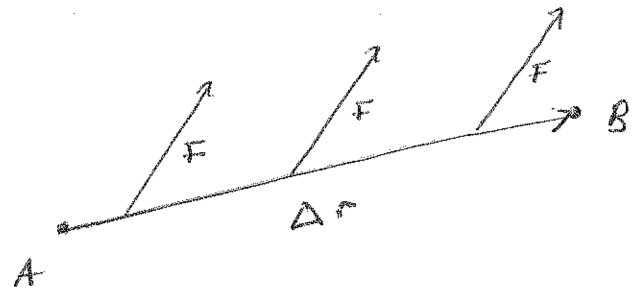
16.2: Line Integrals

From the perspective of work

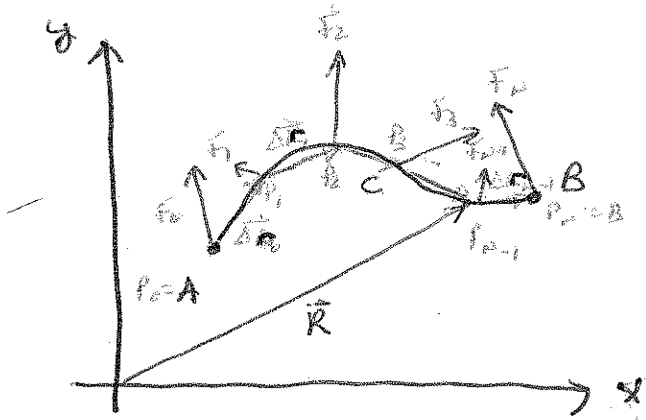


The work to move a particle from A to B by the constant force F is $\omega = F \cdot \Delta r$.

Now suppose that F is not constant and that the particle does not move along a straight line.

$$\vec{F} = \vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$$

and the particle moves along C where C is parametrized by $(x(t), y(t))$ on $t_1 \le t \le t_2$.



$$w \approx \sum_{k=0}^{n-1} \vec{F_k} \cdot \overrightarrow{\Delta r_k}$$

AND

$$\int_C \vec{F} \cdot \overrightarrow{dr} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \vec{F_k} \cdot \overrightarrow{\Delta r_k}$$

so $dW = \vec{F} \cdot d\vec{R}$ and $W = \int dW = \int_C \vec{F} \cdot d\vec{R}$

recall from 16.1 that the force field F is:

$$\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle = M(x,y)\vec{i} + N(x,y)\vec{j}$$

w/ position $\vec{r} = x\vec{i} + y\vec{j}$ and $d\vec{r} = dx\vec{i} + dy\vec{j}$

w/ parametrized curve C $(x(t), y(t))$ on

$t_1 \leq t \leq t_2$

$$\Rightarrow F \cdot d\vec{r} = \langle M(x,y), N(x,y) \rangle \cdot \langle dx, dy \rangle$$

$$= M(x,y)dx + N(x,y)\,dy$$

$$\Rightarrow \text{work: } W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C M(x,y)dx + N(x,y)dy$$

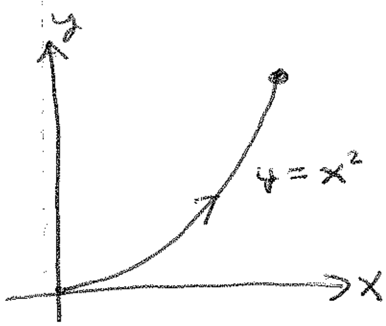
Now parametrize C

$$\Rightarrow \quad W = \int_{t_1}^{t_2} M(x(t), y(t))\frac{dx}{dt}\,dt + N(x(t), y(t))\frac{dy}{dt}\,dt$$

$$= \int_{t_1}^{t_2} \left[M(x,y)\frac{dx}{dt} + N(x,y)\frac{dy}{dt} \right] dt$$

thus we can calculate work w/ a single integral WRT the single variable t.

Ex1: Evaluate $I = \int_C x^2y\,dx + (x-y)\,dy$ where C is the segment of $y=x^2$ from $(0,0)$ to $(1,1)$.



parametrization

$x = t$ & $y = t^2$ on $0 \le t \le 1$

$dx = dt$ $\quad dy = 2t\,dt$

$$I = \int_0^1 t^2 \cdot t^2\,dt + (t - t^2)\,2t\,dt$$

$$= \frac{11}{30}$$

Ex1 rev: Same integral... different parametrization

$x = \sin t$ & $y = \sin^2 t$ on $0 \le t \le \frac{\pi}{2}$

PT: The parametrization doesn't matter so long as the direction stays the same

NOTE: $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$

To be clear, we interpret $I = \frac{11}{30}$ as the work required for a particle to move thru $\vec{F} = \langle x^2y, x-y\rangle$ along $C: y = x^2$ from $(0,0)$ to $(1,1)$.

In the last example, we found the work (evaluated the line integral) by using a parametrization that was somewhat arbitrary in that it doesn't clearly connect t to C (for the pos. vec. $\vec{R}$)

An alternative is think of the position fct $\vec{R}$ as a fct of the arclength s measured from the initial point A.

review sections 13.2 and 13.3.

$\vec{r}(t)$ pos. fct.

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{d\vec{r}}{ds}$ unit tangent vector

Now we can write:

work: $w = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds = \int_C \vec{F} \cdot \vec{T}\, ds$

where $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

That is, the line integral can be thought of as the integral of the tangential component of $\vec{F}$ along the curve C.

Special case: If C is along the x-axis on $a \le x \le b$ and $\vec{F} = f(x)\vec{\imath}$ (the force is parallel to the direction of travel), then $w = \int_a^b f(x)\,dx$

<u>ex2:</u> $w = \int_C (x^2 + y^2 + z^2)\, ds$ where C is the helix $x = t$, $y = \cos 2t$, and $z = \sin 2t$ on $0 \le t \le 2\pi$.

$$x^2 + y^2 + z^2 = t^2 + \sin^2 2t + \cos^2 2t$$
$$= 1 + t^2$$

$$ds = \sqrt{1^2 + (-2\sin 2t)^2 + (2\cos 2t)^2}\, dt$$
$$= \sqrt{5}\, dt$$

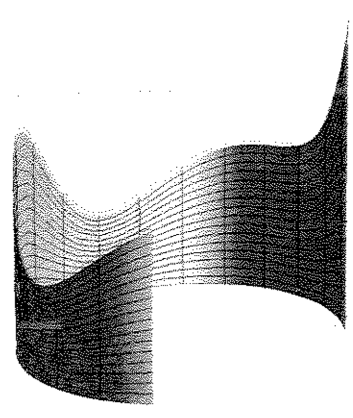
$$\Rightarrow \quad \omega = \int_0^{2\pi} \sqrt{5}\,(1+t^2)\, dt$$
$$= \sqrt{5}\left[t + \frac{t^3}{3}\right]_0^{2\pi}$$
$$= \sqrt{5}\left(2\pi + \frac{8}{3}\pi^3\right)$$

The problem is that ds is difficult to work w/ except in extremely contrived examples. so generally we just find a convenient parametrization of C.

So, the bad news is that the formula w/ ds is tough to work w/. On the other hand, you can visualize $\omega = \int_C f(x,y)\,ds$

$$= \int_{t=a}^{t=b} f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

as the area of the great wall.



Ex3: Evaluate $I = \int_C x^2 y\,dx + (x-y)\,dy$ where C is the straight line segment from $(0,0)$ to $(1,1)$.

This is the same integrand as in ex1.

$$x = t \;\&\; y = t \quad \text{on} \quad 0 \le t \le 1$$

$$\Rightarrow dx = dt \;\&\; dy = dt$$

$$\text{AND} \quad I = \int_0^1 t^2 \cdot t\,dt + (t-t)\,dt = +\frac{1}{4}$$

PT: different paths may lead to different values.

NOTE: In the previous two examples we would say $m(x,y) = x^2y$ & $n(x,y) = x-y$ or we might say the vector field

$$\vec{F} = x^2y\,\vec{i} + (x-y)\,\vec{j}$$

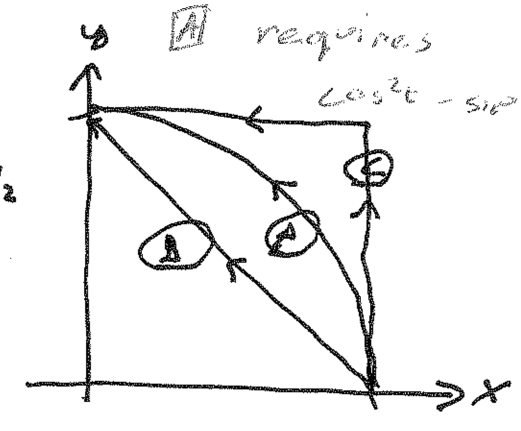
Ex4: Evaluate $I = \int_C y\,dx + (x+2y)\,dy$ over three curves

$\boxed{A}$ requires $\cos^2 t - \sin^2 t = \cos 2t$

$\boxed{A}$ C_A: $(\cos t, \sin t)$ on $0 \le t \le \pi/2$

C_B: $(t, 1-t)$ on $0 \le t \le 1$

$$C_c: \begin{cases} (1, t), & 0 \le t \le 1 \\ (2-t, 1), & 1 \le t \le 2 \end{cases}$$



Ans: 1 in all 3 cases.

<u>PT</u>: All three integrals have the same value & have the same start/end points ... would any curve starting @ (1,0) & ending @ (0,1) have the same value? (see ex 2) ... Yes – for this integral.

This is a foreshadow of cool stuff to come regarding conservative vector fields.
gravitation & electric fields
– magnetic fields are not conservative

Finally, lets talk about closed curves (same start and end). These are noted as $\oint$.

<u>Ex 5</u>: Calculate $\oint \vec{F} \cdot d\vec{R}$ where $\vec{F} = y\vec{i} + 2x\vec{j}$ where C is the unit circle traversed c.c.w

$x = \cos t$ & $y = \sin t$ on $0 \le t \le 2\pi$

$\Rightarrow dx = -\sin t\, dt$ & $dy = \cos t\, dt$

Ans $= \pi$

requires half angle formulas

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

```
function [] = lineintegral(n,opt)
%[sum] = lineintegral(n)
% This primary function calculates the line integral of F = <y,x+2y>
% from (1,0) to (0,1) along the curve C formed by the unit circle in
% the first quadrant

a=0;
b=pi/2;
dt=(b-a)/n;

sum = 0; %initialize the sum

if opt == 1
    'left sum'
    for i=0:n-1
        t = a + i*dt;
        sum = sum + dot(F(R(t)),dR(t));
    end

elseif opt == 2
    'right sum'
    for i=1:n
        t = a + i*dt;
        sum = sum + dot(F(R(t)),dR(t));
    end

elseif opt == 3
    'trapezoidal sum'
    for i=1:n-1
        t = a + i*dt;
        sum = sum + dot(F(R(t)),dR(t));
    end
    sum = 2*sum;
    sum = (sum + dot(F(R(a)),dR(a)) + dot(F(R(b)),dR(b)))/2;

else
    'undefined option'

end

if opt == 1 | opt == 2 | opt == 3
    sum = sum*dt %account for the rectangle widths
end

end

function [force_vector] = F(xy_coordinate) %vector field subfunction

x = xy_coordinate(1);
y = xy_coordinate(2);

P = y;
```

```
Q = x + 2.*y;

force_vector = [P Q];

end

function [xy_coordinate] = R(t) %parametrization subfunction

xy_coordinate = [cos(t) sin(t)];

end

function [dXdY_vector] = dR(t) %differential subfunction

dXdY_vector = [-sin(t),cos(t)];

end
```