16.1: Vector Fields

MATHEMATICA

real world examples.

velocity vector fields (wind & ocean)

force fields (gravitation & magnetic).

Dfn: Let $D \subset \mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a fct $\vec{F}$ that assigns $(x,y) \in D$ a vector $\vec{F}(x,y)$

$$\vec{F} : (x,y) \longmapsto \langle P(x,y),\ Q(x,y) \rangle$$

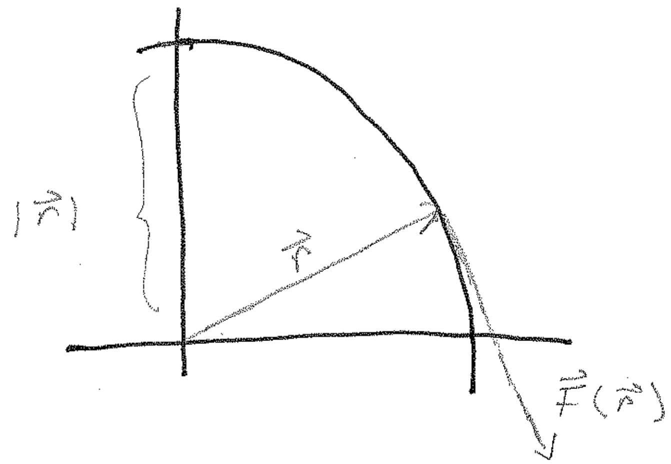
or $F = \langle P, Q \rangle$

in $\mathbb{R}^3$.... $\vec{F} = \langle P, Q, R \rangle$

ex1: sketch $\vec{F} = y\vec{i} + \frac{1}{2}\vec{j}$.

ex2: If $\vec{r} = \langle x, y \rangle$ and $\vec{F}(\vec{r}) = \langle y, -x \rangle$, then $\vec{r} \cdot \vec{F} = 0$. So $\vec{F} \perp \vec{r}$ (the position vector).

To see this, imagine a circle w/radius $|\vec{r}|$.



$\vec{F}$ is $\parallel$ to the tangent vector & its magnitude is :

$$|\vec{F}| = \sqrt{y^2 + (-x)^2}$$
$$= \sqrt{x^2 + y^2}$$
$$= |\vec{r}| \text{ (same as radius)}$$

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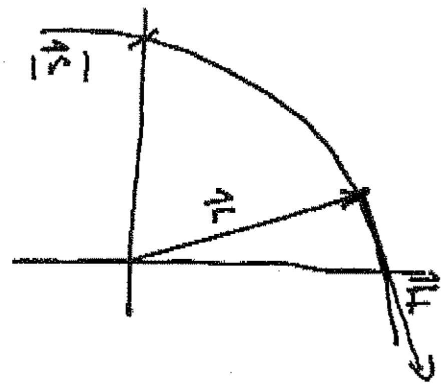
Ex 2: If $\vec{r} = \langle x, y \rangle$ and $\vec{F}(\vec{r}) = \langle y, -x \rangle$ then $\vec{r} \cdot \vec{F} = 0$ .... so $\vec{F}$ is always $\perp$ to its position vector $\vec{r}$... imagine a circle w/ radius $|\vec{r}|$ ....



$\vec{F}$ is $\parallel$ to the tangent vector ... and its magnitude is

$$|\vec{F}| = \sqrt{(y)^2 + (-x)^2}$$
$$= \sqrt{x^2 + y^2}$$
$$= |\vec{r}|$$ the same as the radius.

One vector field we have seen in the past is the gradient field. In $\mathbb{R}^2$

$$f: (x, y) \longmapsto z$$
$$\nabla f: (x, y) \longmapsto \langle f_x(x, y), f_y(x, y) \rangle$$

Ex 3: Plot the gradient field & contour plot together...

(a) $f(x, y) = \sin x + \sin y$

(b) $f(x, y) = \sin(x, y)$

Q: what happens when you go w/ the gradient? Against ... perpendicular

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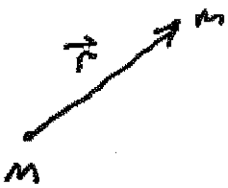
In order to set up a future application (in 16.3), recall from physics that Newton's Law of Gravitation states that the magnitude of the gravitational force between 2 objects w/ masses m & M is



$$\underbrace{|F| = \frac{mMG}{|\vec{r}|^2}}_{\text{scalar.}}$$

... G a gravitational constant

$|\vec{r}|$ is the dist. betw m & M.

If M is @ the origin, then the force $\vec{F}$ is in the direction of $-\vec{r}$, and we can write $\vec{F} = \dfrac{mMG}{|\vec{r}|^2} \cdot \dfrac{-\vec{r}}{|\vec{r}|}$

$$= -\frac{mMG}{r^3} \cdot \vec{r}$$

$\vec{r} = \langle x, y, z \rangle$ (a position vector)

so $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

AND

$$\vec{F}(x,y,z) = \frac{-mMG}{(x^2+y^2+z^2)^{3/2}}\, x\,\vec{i} + \frac{-mMG}{(x^2+y^2+z^2)^{3/2}}\, y\,\vec{j} + \frac{-mMG}{(x^2+y^2+z^2)^{3/2}}\, z\,\vec{k}$$

Now, if $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$, then $\nabla f = \vec{F}$.

... when $\exists f$ s.t. $\nabla f = \vec{F}$, we call $\vec{F}$ a conservative vector field & f its potential fct.

Q: where have you heard the words conservative & potential?