15.6: Triple Integrals

Pretty Basic...

Single integrals — over an interval

double integrals — over a region

triple integrals — over a solid.

Interp ...

Not so basic ... hypervolume?

Fubini's Thm holds (Triple integrals can be written as iterated integrals).

Ex1 (crunchy): $\int_0^{\sqrt{\pi}} \int_0^{x} \int_0^{xz} x^2 \sin y \; dy\, dz\, dx$

$$= \int_0^{\sqrt{\pi}} \int_0^{x} \Big[-x^2 \cos y \Big]_0^{xz} dz\, dx$$

$$= \int_0^{\sqrt{\pi}} \int_0^{x} \left(x^2 - x^2 \cos xz \right) dz\, dx$$

$$= \int_0^{\sqrt{\pi}} \Big[x^2 z - x \sin xz \Big]_0^{x} dx$$

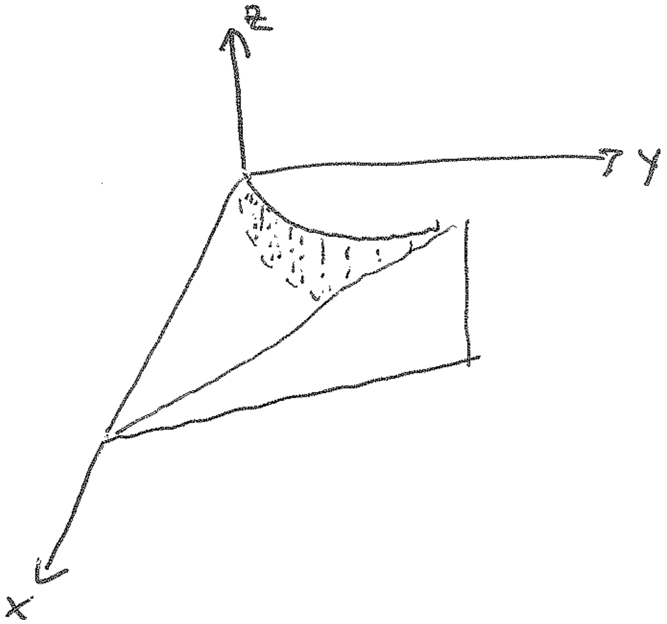
$$= \int_0^{\sqrt{\pi}} \left(x^3 - x \sin x^2 \right) dx$$

$$= \left[\frac{x^4}{4} + \frac{\cos x^2}{2} \right]_0^{\sqrt{\pi}}$$

$$= \left(\frac{\pi^2}{4} + \frac{(-1)}{2} \right) - \left(0 + \frac{1}{2} \right)$$

$$= \frac{\pi^2}{4} - 1$$

Ex 2: Write five other iterated integrals that are equal to $I = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$



See 3D model and Mathematica notebook.

Key:

Innermost variable: arrow parallel to axis

Then project onto the plane of outer 2 variables.

For second/middle variable, arrow parallel to middle variables axis.

(A) $\int_0^1 \int_{\sqrt{y}}^1 \int_0^y f\,dz\,dx\,dy$

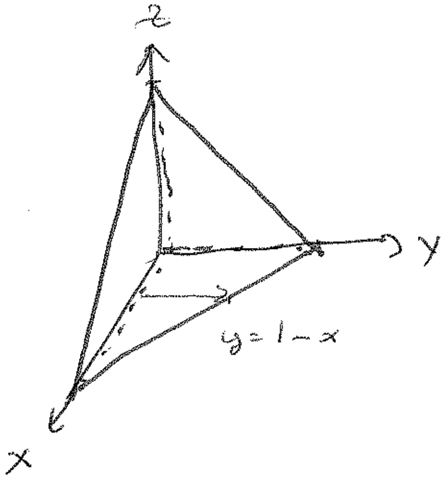
(B) $\int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f\,dy\,dx\,dz$

(C) $\int_0^1 \int_0^{x^2} \int_z^{x^2} f\,dy\,dz\,dx$

(D) $\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f\,dx\,dz\,dy$

(E) $\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f\,dx\,dy\,dz$

Ex 3: Find the mass & center of mass of the tetrahedron bounded by $x=0$; $y=0$; $z=0$; & $x+y+z=1$ w/ density $\rho(x,y,z)=y$.



$$m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \underbrace{y(1-x-y)}_{y - xy - y^2} \, dy\, dx$$

$$= \int_0^1 \left[\frac{y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3} \right]_0^{1-x} dx$$

Let $u = 1-x$
$-du = dx$

$$= \int_0^1 \left(\frac{(1-x)^2}{2} - \frac{x(1-x)^2}{2} - \frac{(1-x)^3}{3} \right) dx$$

$$= -\int_1^0 \left(\frac{u^2}{2} - \frac{(1-u)u^2}{2} - \frac{u^3}{3} \right) du$$

$$= -\int_1^0 \frac{u^3}{6}\, du$$

$$= \left[-\frac{u^4}{24} \right]_1^0$$

$$= \frac{1}{24}$$

$$m = \iiint_E \rho(x,y,z)\, dV$$

$$M_{xy} = \iiint_E z\rho\, dV \qquad M_{xz} = \iiint_E y\rho\, dV \qquad M_{yz} = \iiint_E x\rho\, dV$$

AND $$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

Ans: $\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5} \right)$

$$M_{xy} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{yz^2}{2} \right]_0^{1-x-y} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{y(1-x-y)^2}{2} \, dy \, dx$$

Let $u = 1-x$ $\quad du = -dx$

$$= -\int_1^0 \int_1^u \frac{y(u-y)^2}{2} \, dy \, du$$

$$= -\frac{1}{2} \int_1^0 \int_0^u (u^2 y - 2uy^2 + y^3) \, dy \, du$$

$$= -\frac{1}{2} \int_1^0 \left[u^2 \frac{y^2}{2} - \frac{2}{3} u y^3 + \frac{y^4}{4} \right]_0^u du$$

$$= -\frac{1}{2} \int_1^0 \underbrace{\frac{u^4}{2} - \frac{2}{3} u^4 + \frac{u^4}{4}}_{\frac{1}{12} u^4} \, du$$

$$= -\frac{1}{2} \left[\frac{1}{12} \cdot \frac{u^5}{5} \right]_1^0$$

$$= \frac{1}{2} \left(\frac{1}{60} \right)$$

$$= \frac{1}{120}$$

AND $\bar{z} = \frac{M_{xy}}{m}$ OR $\frac{\frac{1}{120}}{\frac{1}{24}} = \frac{24}{120} = \frac{2}{10} = \frac{1}{5}$