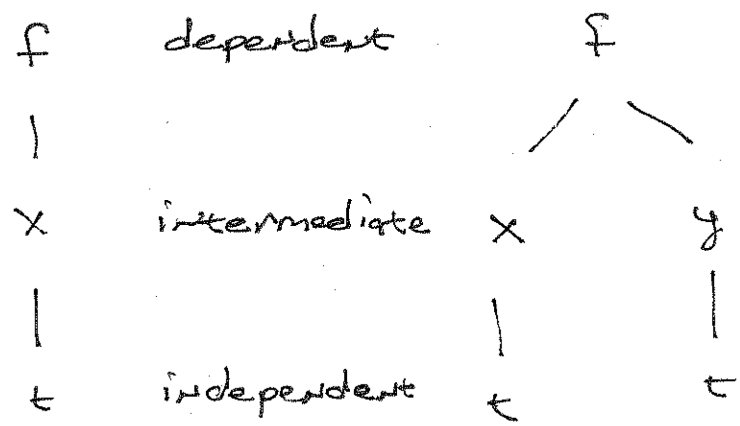
14.5: Chain Rule

recall, if $w(t) = f(x(t))$, then

$$w'(t) = f'(x(t)) \cdot x'(t).$$

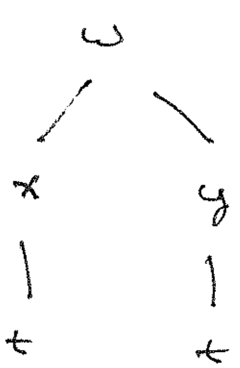
what do we do when $w(t) = f(x(t), y(t))$?



$$\frac{df}{dx} \cdot \frac{dx}{dt} \qquad\qquad \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Ex1: If $w = 3x^2 + 2xy - y^2$

where $x = \cos t$ & $y = \sin t$, find $\frac{dw}{dt}$.



$$\frac{dw}{dt} = (6x + 2y)(-\sin t) + (2x - 2y)\cos t$$

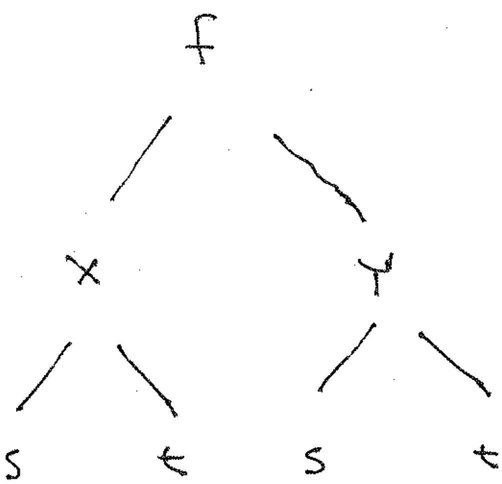
$$= -6\cos t \sin t - 2\sin^2 t + 2\cos^2 t - 2\cos t \sin t$$

$$= -8\cos t \sin t - 2\sin^2 t + 2\cos^2 t$$

you can check by subbing 1st.

14.5
2/4

If $f(\overset{x}{\overset{\shortparallel}{g(s,t)}}, \overset{y}{\overset{\shortparallel}{h(s,t)}})$, how do we find f_s & f_t.



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

ex2: If $z = \cot\left(\frac{v}{u}\right)$ & $u = 2s - 3t$

$v = 5s + t$

Find $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s}$

$$= -\csc^2\left(\frac{v}{u}\right) \cdot \frac{-v}{u^2} \cdot 2 + {}^{-}\csc^2\left(\frac{v}{u}\right) \cdot \frac{1}{u} \cdot 5$$

$$= \frac{2(5s+t)}{2s-3t}\csc^2\left(\frac{5s+t}{2s-3t}\right) - \frac{5}{2s-3t}\csc^2\left(\frac{5s+t}{2s-3t}\right)$$

Find $\frac{\partial z}{\partial t}$ as a HW exercise.

14.5
3/4

Down the road, we will be learning about cylindrical coordinates which will require that we rewrite x & y in terms of r & θ.

Ex 3: Assuming cont. derivs, if $z = f(x, y)$ where $x = r\cos\theta$ & $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$, and $\frac{\partial^2 z}{\partial r \partial \theta}$

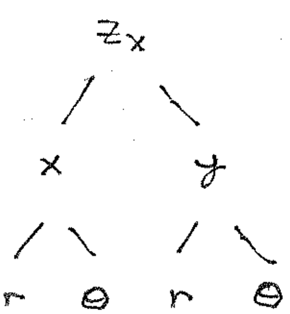
a) $z_r = z_x \cos\theta + z_y \sin\theta$

b) $z_\theta = -z_x r\sin\theta + z_y r\cos\theta$

use the differential operator.

c) $z_{\theta r} = \frac{\partial}{\partial r}\left(-z_x \cdot r\sin\theta + z_y \cdot r\cos\theta\right)$

Warning: this will require two product rules & two chain rules.



$$z_{\theta r} = \left[(z_{xx} x_r + z_{xy} y_r)(-r\sin\theta) + z_x(-\sin\theta)\right]$$
$$+ \left[(z_{yx} x_r + z_{yy} y_r)(r\cos\theta) + z_y \cos\theta\right]$$

$$= -r\sin\theta\cos\theta \, z_{xx} - r\sin^2\theta \, z_{xy} - \sin\theta \, z_x$$
$$+ r\cos^2\theta \cdot z_{yx} + r\sin\theta\cos\theta \, z_{yy} + \cos\theta \, z_y.$$

$$= \cos\theta \, z_y - \sin\theta \, z_x + \frac{r}{2}\sin 2\theta \, (z_{yy} - z_{xx})$$
$$+ r\cos 2\theta \, z_{yx}$$

since $z_{xy} = z_{yx}$ by Clairaut's Thm.

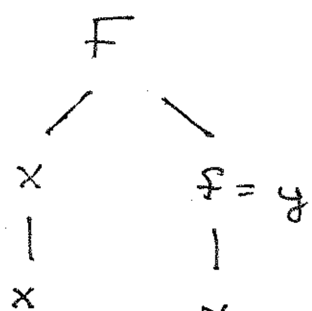
using the chain rule to perform implicit differentiation.

Suppose $F(x,y)=0$ defines y implicitly as a fct of x. (begin ex 4).

That is $y=f(x)$ where $F(x,f(x))=0 \ \forall x \in D_f$.

If f is differentiable, then by the chain rule:

$$0 = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$



solve for $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Ex 4: Find $\frac{dy}{dx}$ if $y^5 + x^2y^3 = 1 + ye^{x^2}$

$$\Rightarrow 0 = 1 - y^5 - x^2y^3 + ye^{x^2} = F(x,y)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{-2xy^3 + 2xye^{x^2}}{-5y^4 - 3x^2y^2 + e^{x^2}}$$