13.3: Arclength, curvature, and the TNB Frame.

I Arclength

$\mathbb{R}^2$: Recall from 10.2 that the arclength of $(x(t), y(t))$ on $a \le t \le b$ is $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

$\mathbb{R}^3$: The arclength of the space curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ on $a \le t \le b$ (assuming a nice curve traversed just once) is:
$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$
$$= \int_a^b |\vec{r}\,'(t)|\, dt$$

ex1: Find the length of $\vec{r}(t) = \langle 3\sin(t), 7t, -3\cos(t) \rangle$ on $-5 \le t \le 5$.

soln.
$$L = \int_{-5}^{5} \sqrt{9\cos^2 t + 49 + 9\sin^2 t}\, dt$$
$$= \int_{-5}^{5} \sqrt{58}\, dt$$
$$= 10\sqrt{58}.$$

Notice that this result is constant, but would vary w/ different limits of integration. This gives us the idea of setting up an arclength fct.

Dfn. If C is a piecewise smooth curve given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ on $a \le t \le b$ and at least one of x, y, z is 1-1 on (a, b), then the arclength fct is:
$$s(t) = \int_a^t |\vec{r}\,'(\underset{\uparrow}{u})|\, du \quad \text{on } a \le t \le b$$
dummy variable

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Differantiating WRT t gives $s'(t) = \frac{ds}{dt} = |\vec{r}\,'(t)|$

key formula

$\frac{ds}{dt} = |\vec{r}\,'(t)|$

where s is the arclength

ex2: Reparameterize $\vec{r}(t) = \langle 3\sin t, 7t, -3\cos t \rangle$ WRT the arclength measured from $(0,0,-3)$ in the direction of increasing t.

soln.

1st: Find t @ $(0,0,-3)$.

By setting coordinates equal, we see $t=0$.

2nd: Relate t and s.

Recall from (ex1) that $|\vec{r}\,'(t)| = \frac{ds}{dt} = \sqrt{58}$

$\Rightarrow s = \sqrt{58}\,t + C$

since the arclength is zero when $t=0$, we know $C=0$.

$\Rightarrow s = \sqrt{58}\,t$

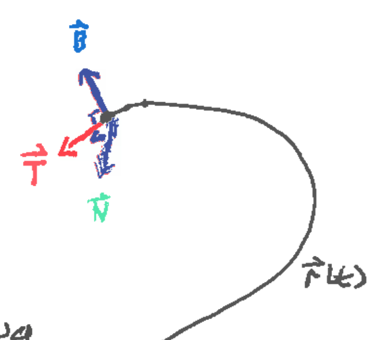
or $t = \frac{s}{\sqrt{58}}$

Thus $\vec{r}[t(s)] = \left\langle 3\sin\left(\frac{s}{\sqrt{58}}\right), \frac{7s}{\sqrt{58}}, -3\cos\left(\frac{s}{\sqrt{58}}\right)\right\rangle$.

Interp: By knowing the distance you have travelled along the curve s, you can know the position in space $\vec{r}$.

II The TNB Frame.

The TNB Frame provides a sense of direction forward/backward left/right up/down for an object (think spacecraft) moving thru space.



we have previously defined the unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Before defining the unit normal vector, we need to prove the following claim.

<u>claim</u>: If $|\vec{r}(t)| = c$ (constant) then $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

$\boxed{\text{Fact \#1}}$

<u>proof</u>.

Suppose $|\vec{r}(t)| = c$

$$\Rightarrow |\vec{r}(t)|^2 = c^2$$

$$\Rightarrow \vec{r}(t) \cdot \vec{r}(t) = c^2$$

differentiate both sides (use the product rule on L.H.S.)

$$\Rightarrow \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\Rightarrow 2\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

Hence $\vec{r}(t) \perp \vec{r}\,'(t)$. QED.

We can apply the previous result as follows: Since $\vec{T}(t)$ is a unit vector, we know $\vec{T}(t) \perp \vec{T}\,'(t)$, so we define the unit normal vector $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$

$\vec{N}(t) \perp \vec{T}(t)$ and in a direction pointed into the curve of $\vec{r}(t)$.

Lastly, we define the binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Key formulas

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ unit tangent vector

$\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ unit normal vector

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ binormal vector

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ex 3 Find $\vec{T}$, $\vec{N}$, and $\vec{B}$ for $\vec{r}(t) = \langle \cos(t), \sin(t)+2, \ln(\cos t)+3 \rangle$ and the osculating (kissing) plane when $t=0$.

soln:

1st: Find $\vec{T}(t)$.

$$\vec{r}'(t) = \left\langle -\sin t, \cos t, \frac{-\sin t}{\cos t} \right\rangle$$

and $|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$

$$= \sqrt{1+\tan^2 t}$$

$$= \sqrt{\sec^2 t}$$

$$= \sec(t) \text{ when } t = 0.$$

so $\vec{T}(t) = \dfrac{1}{\sec(t)} \langle -\sin t, \cos t, -\tan t \rangle$

$$= \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle$$

2nd: Find $\vec{N}(t)$.

$$\vec{T}'(t) = \langle -\cos^2 t + \sin^2 t, 2\sin t \cos t, -\cos t \rangle$$

$$= \langle -\cos 2t, \sin 2t, -\cos t \rangle$$

once you have $\vec{T}'(t)$, it is easiest to evaluate at the given value of $t=0$.

$\Rightarrow \vec{T}'(0) = \langle -1, 0, -1 \rangle$

and $|\vec{T}'(0)| = \sqrt{2}$

so $\vec{N}(0) = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$

and we also have $\vec{T}(0) = \langle 0, 1, 0 \rangle$

3rd: Find $\vec{B}(0)$

$$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

Key formula

$\vec{B}(t) \cdot (\langle x, y, z \rangle - \vec{r}(t)) = 0$

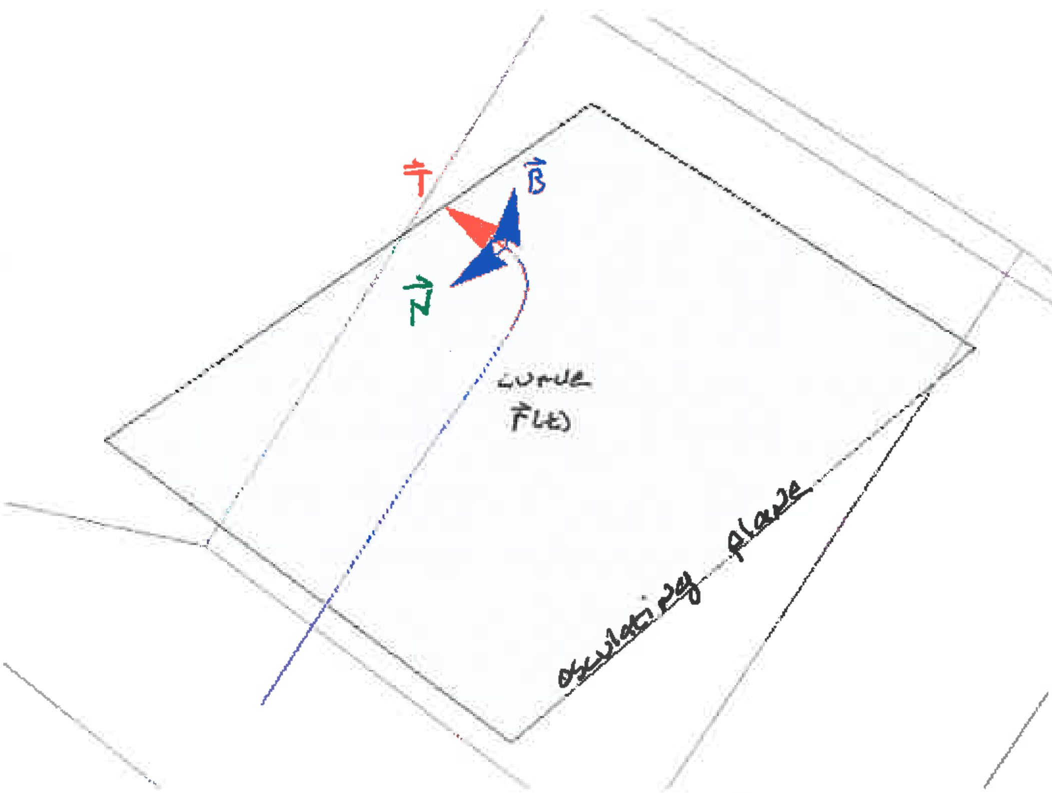
is the eqt. of the osculating plane to $\vec{r}(t)$ @ t.

4th. Find the osculating plane

point: $\vec{r}(0) = \langle 1, 2, 3 \rangle$

Normal vector: $\vec{B}(0) = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$

so the plane is: $-\frac{1}{\sqrt{2}}(x-1) + 0(y-2) + \frac{1}{\sqrt{2}}(z-3) = 0$



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III Curvature

With arclength and the TNB frame in hand, we can now define a new quantity "curvature."

picture how unit tangent vectors would change along equally spaced lengths of curve.

Defn: The curvature κ of a curve is

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| \qquad (\text{defn. } \#1)$$

While an intuitive definition, this is challenging to use when conducting calculations, so we apply the chain rule to derive a more useful formula for κ.

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\frac{dt}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \frac{\vec{T}'(t)}{|\vec{r}'(t)|}$$

and $$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \qquad (\text{defn } \#2)$$

ex 4: Show that the curvature of a circle w/ radius a is $\kappa = \frac{1}{a}$

soln:

First: Note that this makes intuitive sense.

<u>second:</u> set up an eqt for a circle w/ radius a.

$$\vec{r}(t) = \langle a\cos t,\ a\sin t,\ 0\rangle.$$

Note that this example would work in $\mathbb{R}^2$ but I'm doing it in $\mathbb{R}^3$ for consistency.

<u>third.</u> Find k

$$\vec{r}\,'(t) = \langle -a\sin t,\ a\cos t,\ 0\rangle$$

and $|\vec{r}\,'(t)| = a$

$$\Rightarrow \vec{T}(t) = \langle -\sin t,\ \cos t,\ 0\rangle$$

$$\Rightarrow \vec{T}\,'(t) = \langle -\cos t,\ -\sin t,\ 0\rangle$$

and $|\vec{T}\,'(t)| = 1$

Hence $k = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \dfrac{1}{a}$ and we have verified that the curvature of a circle w/ radius a is $\dfrac{1}{a}$.

We already have two formulas for the curvature k but still have two to go. To derive the next formula, we need 4 facts.

<u>Fact #1:</u> If $|\vec{r}(t)| = c$, then $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

<u>Fact #2:</u> $\dfrac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

<u>Fact #3:</u> $\vec{v} \times \vec{v} = \vec{0}$

<u>Fact #4:</u> $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

Now we are ready to derive our formula

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} \Rightarrow \vec{r}\,' = |\vec{r}\,'|\vec{T}$$

$$\Rightarrow \vec{r}\,' = \frac{ds}{dt}\vec{T} \quad \text{(Fact \#2)}$$

Differentiate both sides using the product rule on R.H.S.

$$\Rightarrow \vec{r}\,'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}\,'$$

$$\Rightarrow \vec{r}\,' \times \vec{r}\,'' = \frac{ds}{dt}\vec{T} \times \left(\frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}\,'\right) \quad \text{(Fact \#4)}$$

$$= \frac{ds}{dt}\frac{d^2s}{dt^2}(\underbrace{\vec{T} \times \vec{T}}_{\text{(Fact\#3)}}) + \left(\frac{ds}{dt}\right)^2(\vec{T} \times \vec{T}\,')$$

$$= \left(\frac{ds}{dt}\right)^2(\vec{T} \times \vec{T}\,')$$

$$\Rightarrow |\vec{r}\,' \times \vec{r}\,''| = \left(\frac{ds}{dt}\right)^2 |\vec{T} \times \vec{T}\,'| \quad \text{(Fact \#1)}$$

since $\vec{T} \perp \vec{T}\,'$

$$= \left(\frac{ds}{dt}\right)^2 |\vec{T}||\vec{T}\,'|$$

$$= \left(\frac{ds}{dt}\right)^2 |\vec{T}\,'|$$

Thus
$$|\vec{T}\,'| = \frac{|\vec{r}\,' \times \vec{r}\,''|}{\left(\frac{ds}{dt}\right)^2}$$

$$= \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^2}$$

recall
$$k = \frac{|\vec{T}\,'|}{|\vec{r}\,'|}$$

so
$$k = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} \quad \text{(Dfn \#3)}$$

If you have a plane curve $y = f(x)$

$$k(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \quad \text{(Dfn \#4)}$$

ex 5: Consider the space curve $\vec{r}(t) = \langle 3\cos t, 1+4t, 2+3\sin t\rangle$

Find $\vec{T}, \vec{N}, \vec{B}, k$ and the osculating plane and circle when $t = \frac{\pi}{6}$

key formulas:

$$k = \left|\frac{d\vec{T}}{ds}\right|$$

$$k = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$k = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

and if given $y = f(x)$

$$k = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

soln.

1st: Find $\vec{T}$

$\vec{r}'(t) = \langle -3\sin t, 4, 3\cos t\rangle$

and $|\vec{r}'(t)| = 5$

$\Rightarrow \vec{T}(t) = \left\langle -\frac{3}{5}\sin(t), \frac{4}{5}, \frac{3}{5}\cos t\right\rangle\Big|_{t=\frac{\pi}{6}} \left\langle \frac{-3}{10}, \frac{4}{5}, \frac{3\sqrt{3}}{10}\right\rangle$

2nd: Find $\vec{N}$

$\vec{T}'(t) = \left\langle -\frac{3}{5}\cos t, 0, -\frac{3}{5}\sin t\right\rangle\Big|_{t=\frac{\pi}{6}} \left\langle \frac{-3\sqrt{3}}{10}, 0, \frac{-3}{10}\right\rangle$

and $|T'(\frac{\pi}{6})| = \sqrt{\left(\frac{-3\sqrt{3}}{10}\right)^2 + \left(\frac{3}{10}\right)^2} = \frac{3}{5}$

$\Rightarrow \vec{N}(\frac{\pi}{6}) = \left\langle -\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right\rangle$

3rd: Find $\vec{B}$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{3}{10} & \frac{4}{5} & \frac{3\sqrt{3}}{10} \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{vmatrix} \text{ when } t = \frac{\pi}{6}$$

$$= \left\langle \frac{2}{5}, -\frac{3}{5}, \frac{2\sqrt{3}}{5}\right\rangle$$

4th: Find the osculating plane when $t = \frac{\pi}{6}$.

point: $\vec{r}(\frac{\pi}{6}) = \langle \frac{3\sqrt{3}}{2}, 1 + \frac{4\pi}{6}, 2 + \frac{3}{2} \rangle$

$= \langle \frac{3\sqrt{3}}{2}, 1 + \frac{2\pi}{3}, \frac{7}{2} \rangle$

Normal vector: $\vec{B}(\frac{\pi}{6}) = \langle \frac{2}{5}, -\frac{3}{5}, \frac{2\sqrt{3}}{5} \rangle$

so the plane is: $\frac{2}{5}(x - \frac{3\sqrt{3}}{2}) - \frac{3}{5}(y - (1 + \frac{2\pi}{3})) + \frac{2\sqrt{3}}{5}(z - \frac{7}{2}) = 0.$

5th: Find k when $t = \frac{\pi}{6}$

$$k = \frac{|\vec{T}'(\frac{\pi}{6})|}{|\vec{r}'(\frac{\pi}{6})|} \begin{matrix} \leftarrow \text{see 2nd step} \\ \leftarrow \text{see 1st step.} \end{matrix}$$

$$= \frac{3/5}{5}$$

$$= \frac{3}{25}$$

Key formula

$kiss(\theta) = \vec{r}(t) + \frac{1}{k}\vec{N}(t) + \frac{1}{k}\cos\theta\,\vec{T}(t) + \frac{1}{k}\sin\theta\,\vec{N}(t)$

is the kissing circle to $\vec{r}(t)$ @ time t.

6th: Find the osculating circle when $t = \frac{\pi}{6}$

radius of the circle: $\frac{1}{k} = \frac{25}{3}$

center of the circle: $\vec{r}(\frac{\pi}{6}) + \frac{25}{3}\vec{N}(\frac{\pi}{6})$

circle w/radius $\frac{25}{3}$ on the osculating plane

$\frac{25}{3}\cos\theta\,\vec{T} + \frac{25}{3}\sin\theta\,\vec{N}$

so $kiss(\theta) = \vec{r}(\frac{\pi}{6}) + \frac{25}{3}\vec{N}(\frac{\pi}{6}) + \frac{25}{3}\cos(\theta)\,\vec{T}(\frac{\pi}{6}) + \frac{25}{3}\sin(\theta)\,\vec{N}(\frac{\pi}{6})$

$= \langle \frac{3\sqrt{3}}{2}, 1 + \frac{2\pi}{3}, \frac{7}{2} \rangle + \frac{25}{3}\langle -\frac{\sqrt{3}}{2}, 0, -\frac{1}{2} \rangle +$

$\frac{25}{3}\cos\theta \langle -\frac{3}{10}, \frac{4}{5}, \frac{3\sqrt{3}}{10} \rangle + \frac{25}{3}\sin\theta \langle -\frac{\sqrt{3}}{2}, 0, -\frac{1}{2} \rangle$

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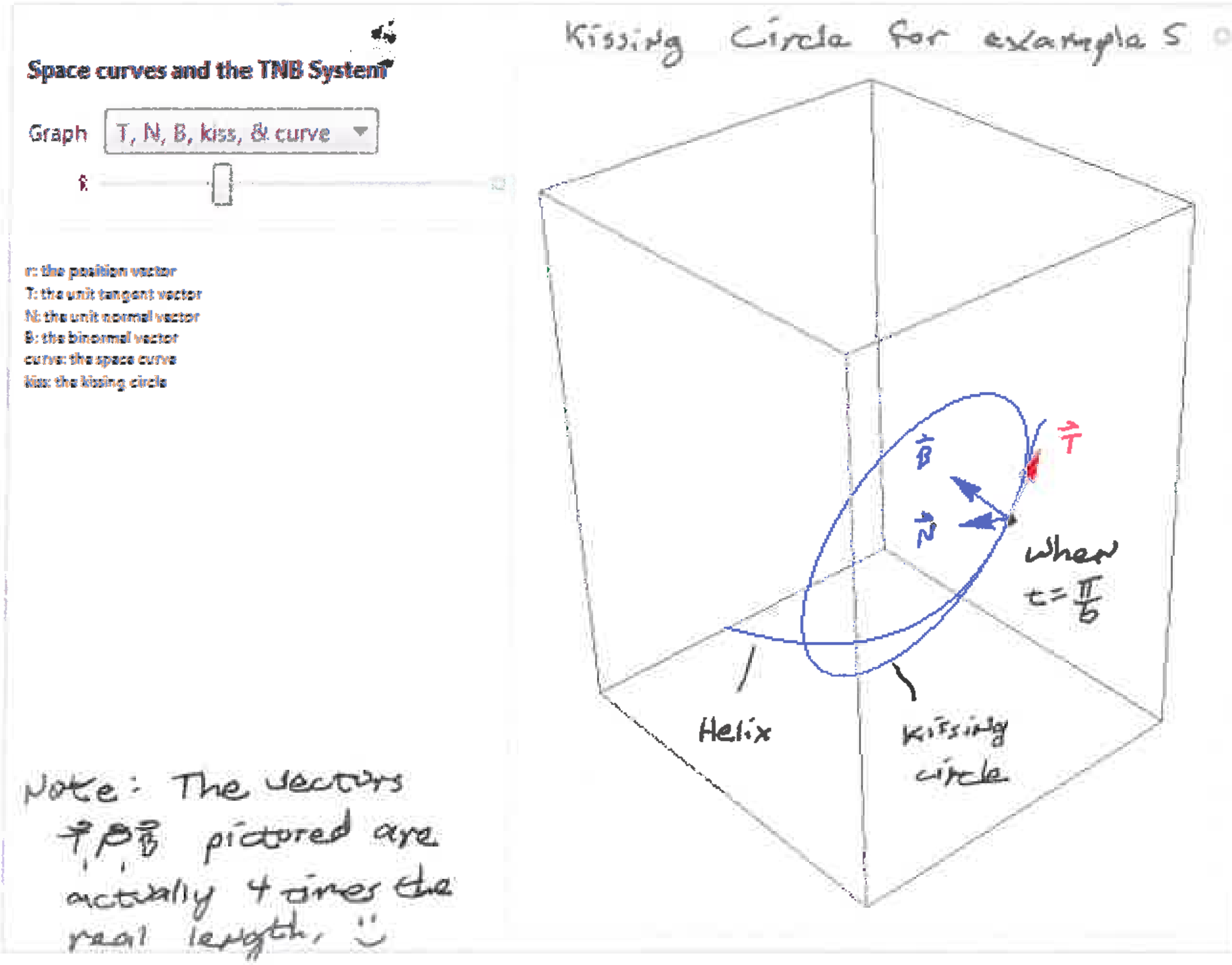

Kissing circle for example 3.