13.2: Derivatives and Integrals of Vector Functions

If $\vec{r}(t)$ is a vector valued fct, then
$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$ provided the limit exists.

Animation of $\vec{r}\,'(t) = \lim_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

Thm: If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ where x, y, z are differentiable fcts, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle$

Ex1: a) Sketch $\vec{r}(t) = \langle 1+t, \sqrt{t}\rangle$. b) Find $\vec{r}\,'(1)$.
c) Find the unit tangent vector $\vec{T}(1)$ when $t=1$.

A curve is smooth if $\vec{r}\,'(t) \neq 0$ (except possibly at endpoints).

Ex 2: Is $r(t) = \langle 4\cos^3 t, 4\sin^3 t\rangle$ smooth?

Thm: Suppose $\vec{u}$ & $\vec{v}$ are vector valued fcts and f is real valued.

a) $\frac{d}{dt}[c\vec{u}(t) \pm \vec{v}(t)] = c\vec{u}\,'(t) \pm \vec{v}\,'(t)$

b) $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

c) $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

d) $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$

e) $\frac{d}{dt}\vec{u}(f(t)) = f'(t)\vec{u}\,'(f(t))$ (chain rule).

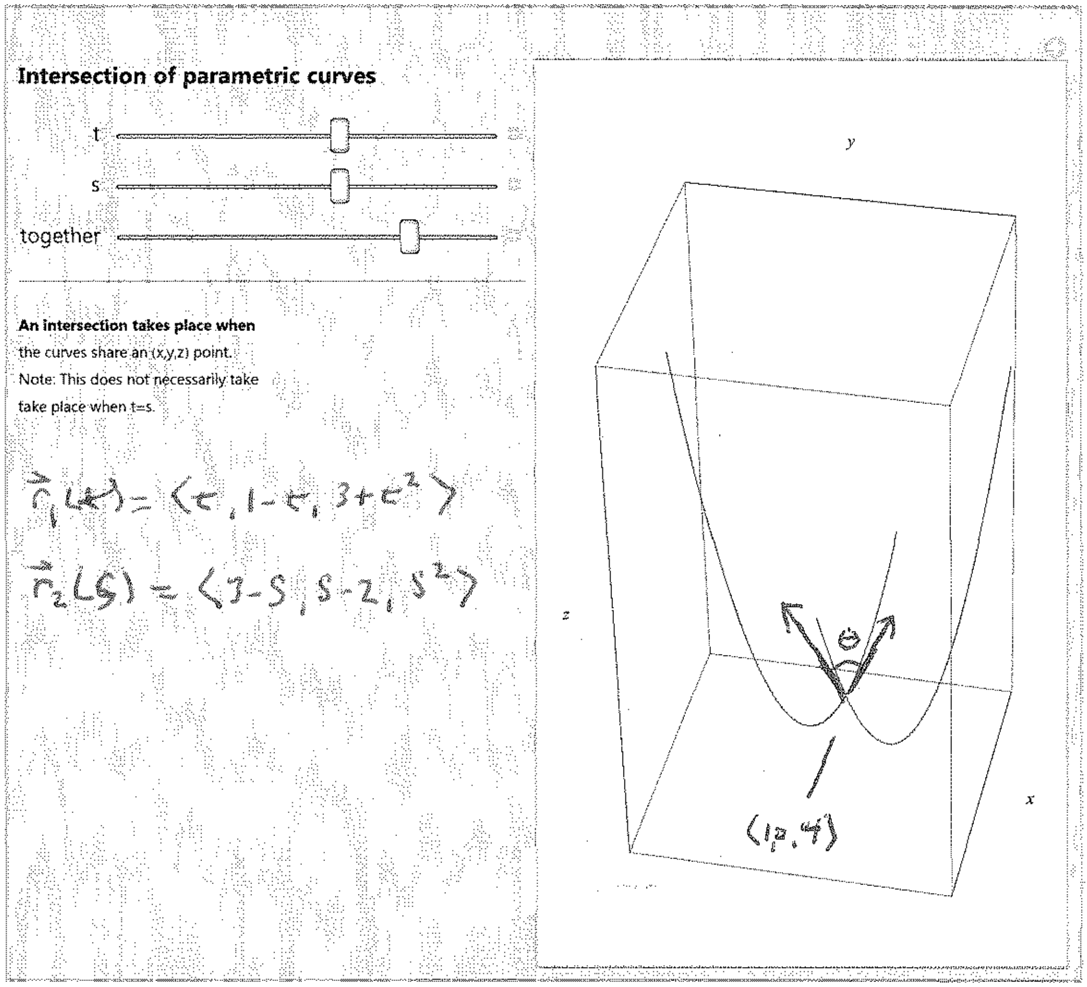
<u>Ex3:</u> At what point do $\vec{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\vec{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$ intersect? Find the angle of intersection.

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

<u>Ex4:</u> $\int \langle e^t, 2t, \ln t \rangle \, dt.$

time permiting, carry on to 13.3.

Out[44]=



Notice that the curves live in the same plane. This is why setting the x or y coordinates equal gives an infinite number of solutions. Setting the z coordinates equal gives $3 + t^2 = s^2$. Now, from the x coordinates, we have that $t = 3 - s$. Squaring both sides gives, $t^2 = (3 - s)^2 = s^2 - 3$ (from the equivalence of the z coordinates). So, $s^2 - 3 = s^2 - 6\,s + 9 \to 6\,s = 12 \to s = 2$. If $s = 2$, then $t = 1$. Now we can find the point of intersection (1, 0, 4) by using the known values for t or s.

To find the angle at the point of intersection, find the tangent vector of each position vector at the appropriate t and s and then use the dot product to find the angle.

we begin by finding tangent vectors @ (1,0,4)

$\vec{r}_1'(t) = \langle 1, -1, 2t \rangle \Rightarrow$ @ $t=1$ we have $\langle 1, -1, 2 \rangle$

$\vec{r}_2'(s) = \langle -1, 1, 2s \rangle \Rightarrow$ @ $s=2$ we have $\langle -1, 1, 4 \rangle$

now we can find the angle w/ $\vec{r}_1' \cdot \vec{r}_2'$

$\langle 1, -1, 2 \rangle \cdot \langle -1, 1, 4 \rangle = |\langle 1, -1, 2 \rangle| |\langle -1, 1, 4 \rangle| \cos\theta$

$\Rightarrow \cos\theta = \dfrac{8}{\sqrt{6 \cdot 18}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{8}{\sqrt{108}}\right) = 0.692$ radians