12.3 : The Dot Product

Defn: If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$. This is called the dot product of $\vec{u}$ and $\vec{v}$.

Ex1: Find $\langle 1, -1 \rangle \cdot \langle -2, 3 \rangle$

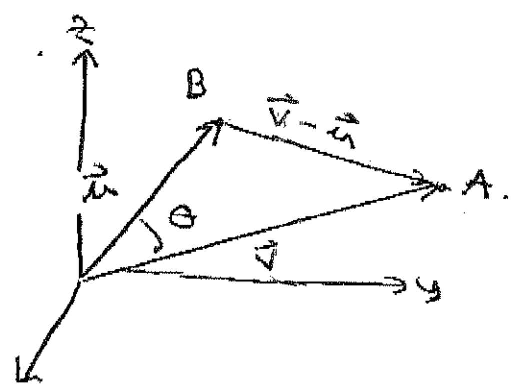
notice that the dot product gives a scalar result. Hence, it is sometimes calle the scalar product.

Ex2: $(5\vec{i} - 8\vec{j} + 13\vec{k}) \cdot (+2\vec{i} + 3\vec{j} - 5\vec{k})$.

Properties of the dot product. $a, b, c \in V_n$ and $c \in \mathbb{R}$

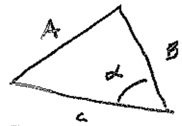
1) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
3) $\vec{0} \cdot \vec{a} = 0$
4) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
5) $(c\vec{a}) \cdot b = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

Geometrically, the dot product is related to the angle between vectors.



Thm: If θ is the angle between $\vec{u}$ and $\vec{v}$, then $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos(\theta)$

<u>proof</u>: use the law of cosines.

recall: 

$B^2 + C^2 = A^2 - 2BC\cos(\alpha)$

$$\Rightarrow |\vec{v}|^2 + |\vec{u}|^2 - 2|\vec{v}||\vec{u}|\cos(\theta) = |\vec{v} - \vec{u}|^2$$

$$= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u})$$

$$= |v|^2 - 2\,\vec{v} \cdot \vec{u} + |u|^2$$

$$\Rightarrow -2|\vec{v}||\vec{u}|\cos\theta = -2\,\vec{v} \cdot \vec{u}$$

$$\Rightarrow \vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}|\cos\theta \quad //.$$

<u>Ex3</u>: If vectors $\vec{u}$ and $\vec{v}$ have lengths 7 and 5 w/ angle between of $\frac{\pi}{4}$, find $\vec{u} \cdot \vec{v}$.

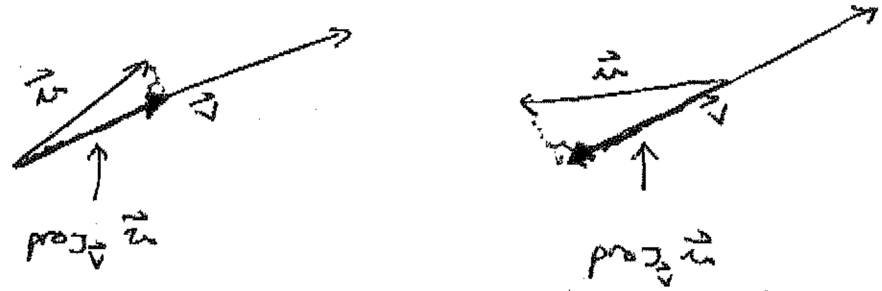
<u>Corollary</u>: If θ is the angle between vectors $\vec{u}$ & $\vec{v}$, then $\cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$.

<u>Ex4</u>: Find the angle between $\langle 1,2,3 \rangle$ and $\langle 4,5,6 \rangle$

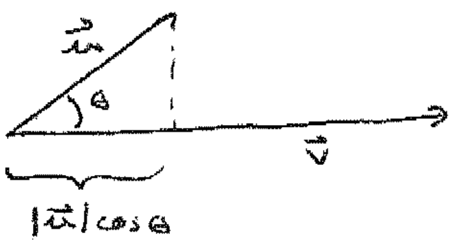
* $\vec{u}$ & $\vec{v}$ are perpendicular (or orthogonal) iff $\vec{u} \cdot \vec{v} = 0$.

Projections

In applications, we frequently project vectors onto each other. For example.



Let's find it... We call the length of $\text{proj}_{\vec{v}}\vec{u}$: $\text{comp}_{\vec{v}}\vec{u}$



So, $\text{comp}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta$

Now consider $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$$= |\vec{v}|\left(|\vec{u}|\cos\theta\right)$$

$$\Rightarrow \quad \text{comp}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

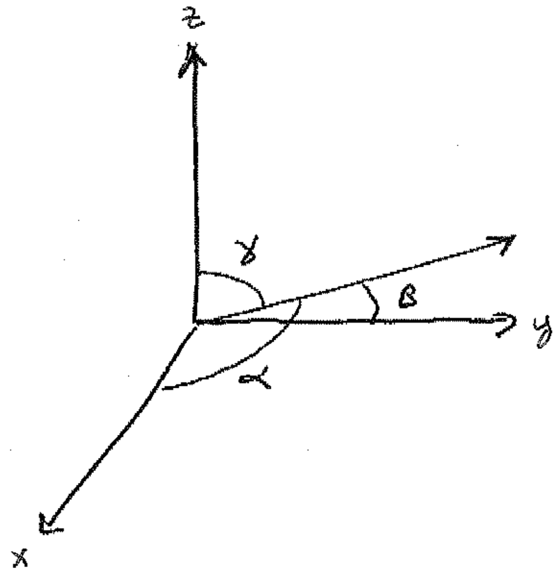
To find $\text{proj}_{\vec{v}}\vec{u}$, multiply the unit vector in the direction of $\vec{v}$ by the length.

$$\text{proj}_{\vec{v}} u = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\cdot\frac{\vec{v}}{|\vec{v}|} = \underbrace{\left(\frac{u\cdot v}{|\vec{v}|^2}\right)}_{\text{scalar.}}\vec{v}$$

Ex 5: Find the projection of $\langle 1,2,3\rangle$ onto $\langle 4,5,6\rangle$

Ex 6: Find the direction cosines by finding the angles between the vector and each axis

use that $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$



so $\vec{u} = |\vec{u}|\langle \cos\alpha, \cos\beta, \cos\gamma\rangle$.

Ex 7: Find the direction cosines of $\langle 1,2,3\rangle$.