11.11 Apps of Taylor Series

<u>Goal</u>: We want to approx a fct w/ a Taylor polynomial of degree n.

<u>ex</u>: $e^x \approx 1 + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$. $n = 4$

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$, $n = 5$

We want this to have a given/known accuracy on a specified interval.

<u>ex</u>: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm 0.0005$ on $|x| < 0.82$

$\uparrow$ Given

<u>scenario 1</u>: Given degree 5, find interval $|x| < 0.82$

<u>scenario 2</u>: Give interval $|x| < 0.82$, find degree 5.

<u>Tools</u>:

<u>Alt. Series Estimation Theorem</u>

If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $b_{n+1} \leq b_n$

(ii) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \leq b_{n+1}$

<u>Taylor's Inequality</u>

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, Then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$ for $|x-a| \leq d$

ex: Use Taylor's Inequality to determine the degree n s.t. $f(x) = e^x = \sum_{n=0}^{m} \frac{x^n}{n!} \pm 0.00005$ on $-2 \le x \le 2$

Given max error

1st: Find M.

$$|f^{(n+1)}(x)| = |e^x| = e^x$$

The max of e^x on $-2 \le x \le 2$ is $M = e^2$.

2nd: set up compound inequality

$$|R_n(x)| \le \frac{e^2}{(n+1)!}|x-0|^{n+1} \le 0.00005 \quad \text{on } |x| \le 2$$

Maclaurin series — Given

$$\Rightarrow \frac{e^2}{(n+1)!}|x|^{n+1} \le 0.00005 \quad \text{on } |x| \le 2$$

3rd: solve worst case scenario (usually w/ table).

$$\frac{e^2}{(n+1)!}2^{n+1} \le 0.00005$$

$\underline{n = 12}$

Calculator table

n	$\frac{e^2\, 2^{n+1}}{(n+1)!}$
11	6.3×10^{-5}
12	9.7×10^{-6}
13	1.4×10^{-6}

4th: state conclusion

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{12}}{12!} \pm 5\times10^{-5}$$

on $|x| \le 2$.

ex: Use the Alt. Series Estimation Thm to determine the degee of the MacLaurin series approx to $f(x) = \sin x$ on $|x| \le -\frac{\pi}{2}$ that is accurate w/in 0.00005.

1st: recognize the alternating series & ID b_{n+1}

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad w/\ b_{n+1} = \frac{|x|^{2(n+1)+1}}{(2(n+1)+1)!} = \frac{|x|^{2n+3}}{(2n+3)!}$$

2nd: set up compound inequality

$$|R_n| \le \frac{|x|^{2n+3}}{(2n+3)!} \le 0.00005$$

3rd: solve worst case scenario (w/table)

$$\frac{\left|\frac{\pi}{2}\right|^{2n+3}}{(2n+3)!} \le 0.00005$$

Calculator table

n	$\frac{(\pi/2)^{2n+3}}{(2n+3)!}$
3	1.6×10^{-4}
4	3.6×10^{-6}
5	5.7×10^{-8}

4th: state conclusion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$