11.4: The Comparison Test

Does $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ converge or diverge?

Well, this reminds us of $\sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a convergent geometric series.

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series w/positive terms.

i.) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

ii.) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

$\square$ proof of (i).

Let $s_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=1}^{n} b_i$, and $t = \sum_{i=1}^{\infty} b_i$

Both sequences $\{a_n\}$ and $\{b_n\}$ are positive so the partial sums are increasing. since $t_n \to t$, so $t_n \leq t$ $\forall n$. Since $a_i \leq b_i$, we have $s_n \leq t_n$. Thus, $s_n \leq t$ $\forall n$. This means $\{s_n\}$ is increasing and bounded above and there converges by the monotonic sequence theorem. $\blacksquare$

The proof of (ii) is in the text.

NOTE: To use the comparison test you must have a known series $\sum b_n$ to compare w/. Generally, we compare using a p-series or a geometric series.

__Ex 1:__ Does $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ converge or diverge?

__Ex 2:__ Does $\sum_{n=1}^{\infty} \frac{\sin^2 n+1}{n}$ converge or diverge?

__Ex 3:__ Does $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converge or diverge?

we do not know using the comparison test...

__The Limit Comparison Test__

Suppose that $\sum a_n$ and $\sum b_n$ are series w/ positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite positive number. Then either both series converge or both diverge.

$\square$ proof.

Let $m, M > 0$ be s.t. $m < c < M$. Because $\frac{a_n}{b_n}$ is close to c for large n, $\exists N$ s.t. $m < \frac{a_n}{b_n} < M$, $n > N$

$$\Rightarrow mb_n < a_n < Mb_n, \quad n > N$$

If $\sum b_n$ converges, so does $\sum Mb_n$ and $\sum a_n$ converges

If $\sum b_n$ diverges, so do $\sum mb_n$ and $\sum a_n$ diverges.

<u>Ex 3 rev:</u> Does $\sum_{n=1}^{\infty} \frac{1}{3^n - 2}$ converge or diverge?

<u>Ex 4:</u> Does $\sum_{n=2}^{\infty} \frac{n^3+2}{n^4-9}$ converge or diverge?

<u>Estimating Sums.</u>

We can estimate the remainder by using our convergent $\sum b_n$ that is above $\sum a_n$. An upper-bound on the remainder of $\sum b_n$ is also an upper-bound on the remainder of $\sum_{i=n}^{\infty} a_i$.

<u>Ex 5:</u> Approximate $\sum_{n=1}^{\infty} \frac{1}{n^5+1}$ to w/in 0.00005 using the least possible terms in the partial sum.... I have an upperbound.

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^5+1}}_{S} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^5}}_{T}$$

for what n is $|T - T_n| \leq 0.00005$ by our Remainder Est. using the integral test.

$$S_n = \sum_{i=1}^{n} \frac{1}{i^5+1} \leq \sum_{i=1}^{n} \frac{1}{i^5} = T_n \quad \text{and}$$

$$\sum_{i=n+1}^{\infty} \frac{1}{i^5+1} \leq \sum_{i=n+1}^{\infty} \frac{1}{i^5} \leq \int_n^{\infty} \frac{dx}{x^5} \leq 0.00005.$$

solve for n.